SSTPRS

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What are symmetries?
Actions that leave something invariant.
E.g. S_3 expressed as rotations and reflections of vertices on a triangle. First exercise: Cayley table of S_3

Homework problem: Cayley table for symmetries of square, and for arbitrarily sided n - gon. Identify operations and how to compose them.

Can consider the same setup in complex plane with roots of unity. Rotation is multiplication by a complex factor, reflection (one of them) is complex conjugation. Compose to get the rest. Note: these are dihedral groups of order $2n$.

Groups where there is a continuum of different ways to change things (e.g. a circle) are Lie Groups, which are related to Lie Algebras. Compare to a discrete group.

Homework problem: the thing with the bugs.

1.1 Generators and stuff

Denote a generator, $\mathbf{L_3}$ such that:

$$\mathbf{L_3}x = -iy$$

$$\mathbf{L_3}y = ix$$

$$\mathbf{L_3}z = 0$$

Which also obeys a Leibniz rule:

$$\mathbf{L_3}(xy) = (\mathbf{L_3}(x))y + x(\mathbf{L_3}(y)) = -i(y^2 - x^2)$$

We can take an exponential using Taylor series:

$$\mathcal{R}_3(\gamma) = e^{-i\gamma\mathbf{L_3}}$$

This gives:

$$\mathcal{R}_3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{R}_3(x) \\ \mathcal{R}_3(y) \\ \mathcal{R}_3(z) \end{bmatrix} = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Which is a rotation about the z axis. We can also similarly define operators $\mathbf{L}_1$ and $\mathbf{L}_2$ that generate rotations about the other two axes. Each of these resulting matrices have determinant 1, and are thus invertible.

These are not group elements, but generators in a Lie Algebra. Commutators are where we get closure, not in multiplication. These are closed under commutation. We have one other requirement for a Lie Algebra: Jacobi Identity:

$$[[L_i, L_j], L_k] + [[L_j, L_k], L_i] + [[L_k, L_i], L_j] = 0$$

Satisfied for any i, j, k, so there are actually 27 things to check here. What we are looking at is $SO(3)$.

Homework problem: Check Jacobi identity for all generators of $SO(3)$.

Homework problem: Write the most general example of a quadratic function and act on it with L_1, L_2, L_3 and see if we notice anything interesting for certain coefficients (where does it vanish?):

$$f(x, y, z) = a_1x^2 + a_2y^2 + a_3y^2 + b_1xy + b_2xz + b_3zy + ...$$

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"Words are nice, math is better"

2.1 Some more notes on generators of $SO(3)$

We did a bunch of calculations on commutators on the L_1, L_2, L_3 generators. Notation:

$$\begin{matrix} \mathbf{L_3}x = -iy & \mathbf{L_2}x = iz & \mathbf{L_1}x = 0 \\ \mathbf{L_3}y = ix & \mathbf{L_2}y = 0 & \mathbf{L_1}y = -iz \\ \mathbf{L_3}z = 0 & \mathbf{L_2}z = -ix & \mathbf{L_1}z = iy \end{matrix}$$

We also found:

$$e^{i\gamma(-\mathbf{L_3})} = e^{-i\gamma\mathbf{L_3}} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{R}_3(\gamma)$$

Which is a matrix that rotates a vector counterclockwise about the z axis. We proved this using Taylor expansions. We can call $\tilde{\mathbf{L}}_\mathbf{3} = -\mathbf{L_3}$, with:

$$\tilde{\mathbf{L}}_\mathbf{3}x = iy$$

$$\tilde{\mathbf{L}}_\mathbf{3} y = -ix$$

So defining similar objects, $\tilde{\mathbf{L}}_\mathbf{1}$ and $\tilde{\mathbf{L}}_\mathbf{2}$ gives commutators:

$$[\tilde{\mathbf{L}}_\mathbf{1}, \tilde{\mathbf{L}}_\mathbf{2}] = i\tilde{\mathbf{L}}_\mathbf{3}$$

$$[\tilde{\mathbf{L}}_\mathbf{2}, \tilde{\mathbf{L}}_\mathbf{3}] = i\tilde{\mathbf{L}}_\mathbf{1}$$

$$[\tilde{\mathbf{L}}_\mathbf{3}, \tilde{\mathbf{L}}_\mathbf{1}] = i\tilde{\mathbf{L}}_\mathbf{2}$$

There is no single "correct" set of operators that give the rotation actions. Notation is arbitrary but must be specified.

2.2 Tensor Notation and Levi-Civita

Take a symbol, and ad a **subscript** or **superscript**. For now:

$$x_1 \equiv x \equiv x^1$$

$$x_2 \equiv y \equiv x^2$$

$$x_3 \equiv z \equiv x^3$$

For example:

$$\vec{V} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow v_1 = 3$$

$$\Rightarrow v_2 = 1$$

$$\Rightarrow v_3 = 2$$

So this is an alternative way of describing vectors with indices. So for two vectors,

$$\vec{W} \cdot \vec{V} = \sum_{i=1}^{3} W_i V_i$$

Another example:

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2}$$

Which using tensors is:

$$m\frac{d^2}{dt^2}x_i = F_i$$

What about a cross product? We use the Levi-Civita symbols:

$$\epsilon_{ij} = i(i-j)$$

$$\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$$

$$\epsilon_{ijkl} = \frac{1}{12}(i-j)(i-k)(i-l)(j-k)(j-l)(k-l)$$

The cross product then becomes:

$$\sum_{i=1}^{3}\sum_{j=1}^{3} W_i V_i \epsilon_{ijk} = T_k$$

Can we use this tensor notation for:

$$[L_1, L_2] = iL_3$$

$$[L_2, L_3] = iL_1$$

$$[L_3, L_1] = iL_2$$

We can write:

$$[L_i, L_j] = i\sum_{k=1}^{3} \epsilon_{ijk} L_k$$

In this case i, j are "free indices", but the ks are "contracted" indices (also known as dummy indices). Einstein summations drop the sum symbol, and we implicitly assume a sum whenever we see pairs of the same (contracted) indices. For example:

$$[L_i, L_j] = i\epsilon_{ijk} L_k$$

Denotes a summation. Another example:

$$\vec{W} \cdot \vec{V} = W_j V_j$$

Another example:

$$\epsilon_{ijk}\epsilon_{ijk}$$

Has three pairs of dummy indices, and hence three sums:

$$\epsilon_{ijk}\epsilon_{ijk} = \sum_i \sum_j \sum_k \epsilon_{ijk}\epsilon_{ijk} = 3! = 6$$

Similarly:

$$\epsilon_{ij}\epsilon_{ij} = \sum_i \sum_j \epsilon_{ij}\epsilon_{ij} = 2! = 2$$

Or in higher dimensions:

$$\epsilon_{i_1,\ldots,i_n}\epsilon_{i_1,\ldots,i_n} = n!$$

Homework Problem:

$$\epsilon_{i_1\ldots i_n}\epsilon_{i_1\ldots i_n} = n!$$

$$\epsilon_{i_1\ldots i_{n-1}k}\epsilon_{i_1\ldots i_{n-1}l} = ?$$

$$\epsilon_{i_1\ldots i_{n-2}k_1k_2}\epsilon_{i_1\ldots i_{n-2}l_1l_2} = ?$$

Bonus:

$$\epsilon_{i_1 \dots i_p k_1 k_2 \dots k_{d-p}} \epsilon_{i_1 \dots i_p l_1 l_2 \dots l_{d-p}} = ?$$

Interesting observation:

$$W_i V_j \delta_{ij} = W_i V_i = \vec{W} \cdot \vec{V}$$

Now consider:

$$\epsilon_{ik_1k_2} \epsilon_{il_1l_2} = \delta_{k_1l_1}\delta_{k_2l_2} - \delta_{k_1l_2}\delta_{k_2l_1}$$

We say something is **antisymmetric** if:

$$A_{ab} = -A_{ba}$$

And something is **symmetric** if:

$$S_{ab} = S_{ba}$$

We can construct new objects:

$$T_{[ab]} = T_{ab} - T_{ba}$$

$$T_{(ab)} = T_{ab} + T_{ba}$$

Example:

$$T_{ab} = X_a Y_b$$

$$T_{[ab]} = X_{[a} Y_{b]} = X_a Y_b - X_b Y_a$$

Homework Problem: Calculate:

$$\mathcal{M}_{1i} \mathcal{M}_{2j} \epsilon_{ij}$$

Where $\mathcal{M}$ is a 2×2 matrix. What about:

$$\mathcal{M}_{1i_1} \mathcal{M}_{2i_2} \dots \mathcal{M}_{ni_n} \epsilon_{i_1 i_2 \dots i_n}$$

2.3 Other stuff I guess

The 3d rotation matrices are given as:

$$R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_2(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_3(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the trace of all of these is the same:

$$\mathrm{Tr}(R_n(a)) = 1 + 2\cos a$$

The 2d rotation matrix is:

$$R(\chi) = \begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix}$$

In 4d we have 6 rotation matrices. They have the form:

$$R(\Theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Theta & -\sin\Theta \\ 0 & 0 & \sin\Theta & \cos\Theta \end{bmatrix}$$

Note that in general:

$$\det(R(a)) = 1$$

Homework Problem: Find the generators for 4 dimensional rotation, and (using computational tools) find the commutation relations. And then the other thing Konstantinos said hopefully someone else wrote it down (find relationship with Levi-Civita or something?). i did not hear him :(

2.4 Spinors and the Pauli matrices

If you do a rotation of the object by 2π radians it doesn't come back to itself. Consider everybody's favorite matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The commutators are given by:

$$[\hat{\ell}_i, \hat{\ell}_j] = i\epsilon_{ijk}\hat{\ell}_k$$

Where:

$$\ell_1 = \frac{1}{2}\sigma_1$$

$$\ell_2 = \frac{1}{2}\sigma_2$$

$$\ell_3 = \frac{1}{2}\sigma_3$$

Note: the square of any Pauli matrix is the identity. Also note that the commutator algebra of the ℓs is isomorphic to that of our Ls from earlier. We can then find the commutator relations of the Paul matrices:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

We can also find that:

$$\sigma_{(i}\sigma_{j)} = 2\delta_{ij}\mathbb{I}$$

Exponentiation generates rotations:

$$e^{i\gamma\hat{\sigma}_3} = \cos\left(\frac{1}{2}\gamma\right)\mathbb{I} + \sin\left(\frac{1}{2}\gamma\right)\sigma_3$$

This object does **not** return to itself after a rotation of 2π. This is due to the factor of $\frac{1}{2}$ which is required for there to be an isomorphism.

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3.1 Homework discussion

Levi-Civita symbol:

$$\epsilon_{i_1 i_2 \dots i_n} = \mathrm{sgn}(P)$$

Where P is the permutation corresponding to the permutation of the indices. Also defined to be zero whenever the indices give an invalid permutation (i.e. symbols repeat).

Antisymmetrization of many indices:

$$T_{[a_1 a_2 \dots a_n]} = \sum_{P \in S_n} \mathrm{sgn}(P) \cdot T_{a_{P_1} a_{P_2} \dots a_{P_n}}$$

Similarly:

$$T_{(a_1 a_2 \dots a_n)} = \sum_{P \in S_n} T_{a_{P_1} a_{P_2} \dots a_{P_n}}$$

A fun (but not required) calculation:

$$\mathcal{M}_{i_1 j_1}\mathcal{M}_{i_2 j_2} \dots \mathcal{M}_{i_n j_n}\epsilon_{j_1 j_2 \dots j_n}$$

"Let's do the Nike method" — Konstantinos (Just Do It)

Finding the generators from the rotation matrices: Taylor expand matrices to first order, set equal to $e^{i\theta L}$, and Taylor expand that. Set terms equal to each other, generator follows from there. I.e. the generator is what corresponds to a very small angle (first order term of Taylor expansion).

3.2 Rotations in 3 dimensions

In 3d let:

$$L_i \equiv a\epsilon_{ijk}\check{L}_{jk}$$

Then consider:

$$\begin{aligned}\epsilon_{lmi}L_i = a\epsilon_{lmi}\epsilon_{ijk}\check{L}_{jk} &= a[\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}]\check{L}_{jk} \\ &= a[\delta_{li}\delta_{mk}\check{L}_{jk} - \delta_{lk}\delta_{mj}\check{L}_{jk}] \\ &= a[\check{L}_{lm} - \check{L}_{ml}] = 2a\check{L}_{lm}\end{aligned}$$

So we have:

$$\check{L}_{lm} = \frac{1}{2a}\epsilon_{lmi}L_i$$

We can consider the commutators:

$$[\check{L}_{pq}, \check{L}_{rs}] = c(\delta_{[p[r}\check{L}_{q]s]}) = c(\delta_{pr}\check{L}_{qs} - \delta_{qr}\check{L}_{ps} - \delta_{ps}\check{L}_{qr} + \delta_{qs}\check{L}_{pr})$$

Where c is just some constant based on what convention we are using. We can also generalize this to d dimensions with:

$$\check{L}_{lm} = c(\epsilon_{lmp_1p_2\dots p_{d-2}}L_{p_1p_2\dots p_{d-2}})$$

3.3 Rotations about an arbitrary axis

We know that L_1, L_2, L_3 describe rotations about x, y, z, but what if we want a rotation about an arbitrary other axis? There is a formula that does this (which we will do only in 3 dimensions).

"The price we pay for simplicity is that the formula is more complicated" — Jim

Let $\hat{n}$ be the axis about which we want to rotate, and angle ψ be the angle about that axis (i, j denotes entries in matrix):

$$\mathcal{R}(\psi, \theta, \phi) = e^{i\psi\hat{n}^iL_i}$$

$$\begin{aligned}\mathcal{R}_{ij}(\psi, \theta, \phi) = \cos\psi\delta_{ij} + (1-\cos\psi)\hat{n}_i\hat{n}_j - \sin\psi\epsilon_{ijk}\hat{n}^k \\ \hat{n}^1 = \cos\phi\sin\theta \\ \hat{n}^2 = \sin\phi\sin\theta \\ \hat{n}^3 = \cos\theta\end{aligned}$$

So note that we can sum up the diagonals to get the trace:

$$\mathrm{Tr}(\mathcal{R}_{ij}) = 1 + 2\cos\psi$$

Homework problem: find the analogue of the second line in d dimensions:

$$\tilde{\mathcal{R}}(\alpha, \beta, \gamma) = e^{i(\alpha L_1 + \beta L_2 + \gamma L_3)} = e^{i\alpha_i L_i}$$

3.4 Pauli matrices

We have 3 Pauli matrices and the identity. Some things to note:

$$\mathrm{Tr}(\mathbb{I}\sigma) = 0$$

$$\mathrm{Tr}(\sigma_i \sigma_j) = 2\delta_{ij}$$

So thinking if the trace like an inner product, the Pauli matrices are orthogonal and span the 2×2 matrices. So we can write:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}\mathrm{Tr}(IM)I + \frac{1}{2}\mathrm{Tr}(\sigma_1 M)\sigma_1 + \frac{1}{2}\mathrm{Tr}(\sigma_2 M)\sigma_2 + \frac{1}{2}\mathrm{Tr}(\sigma_3 M)\sigma_3$$

We can use some tensor notation. For example: $(\sigma^i)_{AB}$ means the ith Pauli matrix, entry AB. For example:

$$(\sigma^1)_{AA} = 0$$

$$\mathrm{Tr}(I\sigma^1) = (I)_{AB}(\sigma^1)_{BA}$$
$$\mathrm{Tr}(\sigma^i\sigma^j) = (\sigma^i)_{AB}(\sigma^j)_{BA}$$

We can write the same equation as above in this new notation:

$$(M)_{AB} = \frac{1}{2}[I_{CD}M_{DC}]I_{AB} + \frac{1}{2}[(\sigma^1)_{CD}M_{DC}](\sigma^1)_{AB} + \frac{1}{2}[(\sigma^2)_C D M_{DC}](\sigma^2)_{AB} + \frac{1}{2}[(\sigma^2)_{CD}M_{DC}](\sigma^2)_{AB}$$

Now let:

$$M_{AB} \equiv U_A V_B$$

We can then differentiate with respect to these variables:

$$\frac{\partial}{\partial U^E}\frac{\partial}{\partial V^F}M_{AB} = \frac{\partial}{\partial U^E}\frac{\partial}{\partial V^F}U_A V_B = \delta_{EA}\delta_{FB}$$

3.5 Groups and Lie Groups

A **group** is a set equipped with a binary operation that satisfies the following:

1. $\forall a, b \in G,\ a \cdot b \in G$
2. $\forall a, b, c \in G,\ (a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. $\exists e \in G$ such that $\forall a \in G,\ a \cdot e = e \cdot a = a$
4. $\forall a \in G,\ \exists a^{-1}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$

Some examples: S_3, $\mathbb{Z}$, invertible square matrices, etc. We can also talk about continuous groups, such as the rotation groups. For a continuous group, we need to provide some type of parameters. For example, rotations in three dimensions require 3 angles (i.e. parameters):

$$G = \{g(\alpha_1, \alpha_2, \ldots, \alpha_n)\}$$

We generally assign:

$$g(0, 0, \dots, 0) = e$$

A **Lie Group** is a group whose elements are parametrized by continuous variables. Each group element corresponds to a point on some smooth manifold. Another example: unitary matrices. Lie groups are at the intersection of group theory and geometry. Another example: $\mathbb{R}$ is a Lie group with the associated manifold being the number line.

Now suppose we have some element of a Lie group, where I takes values from 1 to N (depending on the number of parameters we have). Because we are on a smooth (differentiable) manifold, we can expand with a Taylor series:

$$g(\alpha^I) = g(0) + \alpha^I T_I + \alpha^I \alpha^J T_{IJ} + \dots$$

Note that $g(0) = e$. The first order term of the expansion is called the generator. Now let:

$$g(\beta^I) = e + \beta^I T_I + \beta^I \beta^J T_{IJ} + \dots$$

Note that the coefficients here are the same, all that is changed is the β^I terms. Because we have a group, we can multiply the elements:

$$g(\alpha^I) \cdot g(\beta^I) = g(h^I(\alpha, \beta))$$

Where h is some function of α, β. For the rotation groups this function is addition, but that isn't necessarily always the case.

Consider the following case:

$$h^I(0, \beta) = \beta^I$$

Since $\alpha(0) = e$. Similarly:

$$h^I(\alpha, 0) = \alpha^I$$

In order to do this Taylor expansion, we made the assumption that our parameters were 'near zero'. There are some elements of some Lie groups where we cannot do such a Taylor expansion. This is largely determined by the topology of the associated manifold. We can consider $h^I(\alpha, \beta)$ to see if we can Taylor expand it similarly:

$$h_I(\alpha, \beta) = 0 + \alpha_I + \beta_I + \alpha^J \beta^K C_{IJK} + \dots$$

1 We can also write:

$$g(\alpha) \cdot g(\beta) = g(h(\alpha, \beta)) = e + h^I(\alpha, \beta) T_I + h^I(\alpha, \beta) h^J(\alpha, \beta) T_{IJ} + \dots$$

Exercise: compare this to the explicitly calculation of Taylor series of h and see what happens. Compare zeroth, first, second order terms.

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4.1 Notes on homework

$SO(3)$ is the group of orthogonal matrices of determinant 1 that preserve length of a vector.

Note: an **orthogonal matrix** is a matrix M such that $M^T M = I$. Thus, they must have determinant ± 1.

The group of all orthogonal matrices, $O(2)$ is also a group. The real Pauli matrices are in $O(3)$, but not $SO(2)$.

All rotations in $SO(3)$ are continuously connected to the identity, but not $O(3)$. We can also extend this to complex vectors with unitary matrices; note that the concept of orthogonal matrices only makes sense for real vector spaces.

Note: come back to problem sent on element.

4.2 Back to Baker-Hausdorff-Campbell and Lie Algebras

Recall: we have group elements parametrized by a single variable, which we then Taylor expanded.

$$g(\alpha^I) = g(0) + \alpha^I T_I + \alpha^I \alpha^J T_{IJ} + \dots$$

$$g(\beta^I) = g(0) + \beta^I T_I + \beta^I \beta^J T_{IJ} + \dots$$

What is the "addition" described here, since it is clearly not the same as the group operation? More formally, we are considering matrix representations of group elements. Let:

$$\Pi(g(\alpha^I)) \ , \ \Pi(g(\beta^I))$$

be these matrix representations. So we inherit the algebraic structure of matrices, allowing for addition and multiplication by scalars. Every Lie group has a matrix representation, as long as we allow for infinite dimensional matrices. We generally just assume that all group elements can be represented as matrices and go from there. So we are really doing a Taylor expansion of matrices. Recall from last time:

$$g(\alpha) \cdot g(\beta) = g(h(\alpha, \beta)) = I + h^I(\alpha, \beta) T_I + h^I(\alpha, \beta) h^J(\alpha, \beta) T_{IJ} + \dots$$

Remember that each group element corresponds to a point on some manifold, with the identity corresponding to a rotation by zero degrees. The T_I coefficients live on the tangent space to the manifold. The index I corresponds to the number of parameters, and thus the dimension of the surface of the manifold, and also gives the number of generators.

Now we can set the above term for $g(\alpha)\cdot g(\beta)$ equal to the explicit multiplication of the two. We then set the terms equal. The second order terms give:

$$\alpha_I \beta_J T_I T_J = \alpha^J \beta^K C^I_{JK} T_I + \alpha^I \beta^J T_{IJ} + \beta^I \alpha^J T_{IJ}$$

Note that T_{IJ} must be symmetric when switching I, J. Hence:

$$\alpha^I \beta^J T_I T_J = \alpha^J \beta^K C^I_{JK} + 2\alpha^I \beta^J T_{IJ}$$

"I am my wife's project, and the project is to make I me civilized" — Jim

We can solve the above for T_{IJ} to give:

$$T_{IJ} = \frac{1}{2}\left(T_I T_J - C^K_{IJ} T_K\right)$$

So the second order matrix, depends only on the first order matrix coefficient, and the function h which is determined by the group operation. We can note that $T_{IJ} = T_{JI}$, and subtract to get:

$$[T_I, T_J] = f^K_{IJ} T_K$$

This is called a **Lie Algebra**, and the f factors are **structure constants**. In the group of rotations, the structure constants are given by the Levi-Civita symbols. So these T_is are generators, since they generate the entire series. Note the example we saw with the rotation group with the L generators.

However there is more to this! Recall the associativity of the group:

$$(g(\alpha)\cdot g(\beta))\cdot g(\gamma) = g(\alpha)\cdot(g(\beta)\cdot g(\gamma))$$

We can use this to show that the Jacobi identity holds (it sounds like a pain to actually do though). A note on conventions: I can multiply each term by a factor of $1 = (-i)i$, creating a "physicists' convention" with:

$$T^{(P)}_I = iT^{(M)}_I$$

This changes the above rule to:

$$[T^{(P)}_I, T^{(P)}_J] = i f^K_{IJ} T^{(P)}_K$$

We can actually generate the entire series with:

$$g(\alpha^I) = e^{-\alpha^I T_I}$$

Again, note the example of the rotations in 3 dimensions.

When we say **Lie Algebra** we mean a set of objects:

$$T_I\ ,\ I = 1, 2, \ldots, n$$

Along with an operation called the **Lie Bracket** satisfying:

$$[T_I, T_J] = i f_{IJ}^K T_K$$

As well as the **Jacobi Identity**:

$$[T_I, [T_J, T_K]] + [T_J, [T_K, T_I]] + [T_K, [T_I, T_J]] = 0$$

When the objects in question are matrices, the Lie Bracket is generally the commutator.

We can then plug in the commutator relations to the Jacobi identity:

$$i^2 \left(f_{JK}^L f_{IL}^P - f_{IK}^L f_{JL}^P + f_{IJ}^L f_{KL}^P \right) T_P = 0$$

But note that the T_Ps are generators and thus linearly independent. So all of the coefficients must equal zero. Rearranging this equation and "massaging" it a bit gives some funny stuff:

$$(t_I)_K^L \equiv -i f_{IK}^L$$

$$(t_I t_J)_K^P - (t_J t_I)_K^P = i f_{IJ}^R (t_R)_K^P$$

And so:

$$([t_I, t_J])_K^P = i f_{IJ}^R (t_R)_K^P$$

And so there is a particular matrix representation that obeys the same relations as the Lie Algebra. This is called the **Adjoint Representation**. If there are n generators, then the adjoint representation is of $n \times n$ matrices. For example: the 2×2 Pauli matrices form a Lie algebra, which has an adjoint matrix representation consisting of 3 matrices of size 3×3.

Homework Problem: Find the adjoint representation for the half Pauli matrices, $SO(3)$, $SO(2)$.

Closure of generators under commutation does not guarantee that something is a Lie algebra. We also need the Jacobi identity.
young tableaux–boxes for combinatorics
the 8 3x3 analogue for pauli SU(2): gell-mann lambda matrices SU(3)

4.3 $SU(3)$ and the Gell-Mann Lambda Matrices

The **Gell-Mann Lambda Matrices** are a set of 8 Hermitian matrices that form a Lie Algebra. They are traceless, but their squares have a trace of 2. They span $SU(3)$.

We can fill in the matrices in the adjoint representation according to:

$$F_I = -i f_{KL}^L$$

Using the given tables. The entry $[ijk]$ corresponds to the j, k entry of matrix i. Also note that we can permute the indices and add negative signs. Note that the first index corresponds to row, the second to column. These tables also give the commutators:

$$[\lambda_i, \lambda_j] = 2i[ijk]\lambda_k$$

Using the same notation in the tables as above.

Group $\rightarrow$ Lie Group $\rightarrow$ Lie Algebra $\rightarrow$ Representation

Remember: the **Lie group** is the operations, the **Lie algebra** is the generators and their relationship. A **representation** is an assignment that maps every group element to a matrix that acts on some vector space. For example, we derived a matrix representation of rotation in 2 and 3 dimensions.

Taylor expanding a group element about the identity will give a matrix representation of a generator. These generators give the structure of the Lie algebra. Exponentiating these generators gives the matrix representations of group elements. Also note that the commutation relations of the generators is also satisfied by their matrix representation:

$$[T_I, T_J] = if_{IJ}^K T_K$$

$$[R(T_I), R(T_J)] = if_{IJ}^K R(T_K)$$

Note that in the second equation, the Lie bracket is precisely the matrix commutator.

Another example $SU(2)$. Special (determinant 1) unitary 2×2 matrices.

Now suppose I have some arbitrary representation of a Lie Algebra:

$$[t_I, t_J] = if_{IJ}^K t_K$$

Then we can complex conjugate to get:

$$[t_I^\star, t_J^\star] = -if_{IJ}^K t_K^\star$$

Note that the fs are assumed real in this convention. Then:

$$[-t_I^\star, -t_J^\star] = if_{IJ}^K (-t_K^\star)$$

And so we have a new representation, called the **complex** representation:

$$T_I = -t_I^\star$$

Homework problem: do this for the Lie algebras we have seen before.

We can follow the same process with transposition, to get another representation:

$$[-t_I^T, -t_J^T] = i f_{IJ}^K (-t_K^T)$$

This is called the **dual** representation. We can do another iteration, taking the Hermitian conjugate, giving a **complex dual** representation, given as the Hermitian conjugate of the original representation.

Some of our favorite Lie groups:

$$O(n, \mathbb{R}) = \{O \in M_{n\times n} \mid O^T O = \mathbb{I}\}$$

$$SO(n, \mathbb{R}) = \{O \in O(n, \mathbb{R}) \mid \det(M) = 1\}$$

5 June 5

"Before we start I'm gonna go to my other office. You all don't know about my other office. This office has my name on it, my other office just has a three letter name on it: 'Men's room'" — Jim

"Being a theoretical physicist is like the worst homework algebra you've ever heard of" — Jim

Consider the first order expansion:

$$\det(I - i\varepsilon H)$$

We then proved explicitly that:

$$\det(I - i\varepsilon H) = 1 - i\varepsilon \mathrm{Tr}(H)$$

Using lots of deltas.

5.1 Back to $SU(3)$

Remember that we had a set of 8 3×3 matrices, $\lambda_1, \ldots, \lambda_8$. A note: Pauli matrices do NOT generate $SU(2)$, you need half of them. And so half of the Gell-Mann matrices generate $SU(3)$. Recall the table of structure constants from last time. We also have:

$$\{\lambda_i, \lambda_j\} = \frac{4}{3}\delta_{ij} I + 2 d_{ijk}\lambda_k$$

We have a similar table for the non vanishing values of d_{ijk} denoted as (ijk). The generators of $SU(3)$ are:

$$F_i = \frac{1}{2}\lambda_i$$

Consider the eigenvalues of λ_3, λ_8 which are the only diagonal matrices in the set. These form the maximal commuting subset. Because they commute they share eigenvalues, and we name the eigenvectors according to these eigenvalues:

$$|\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

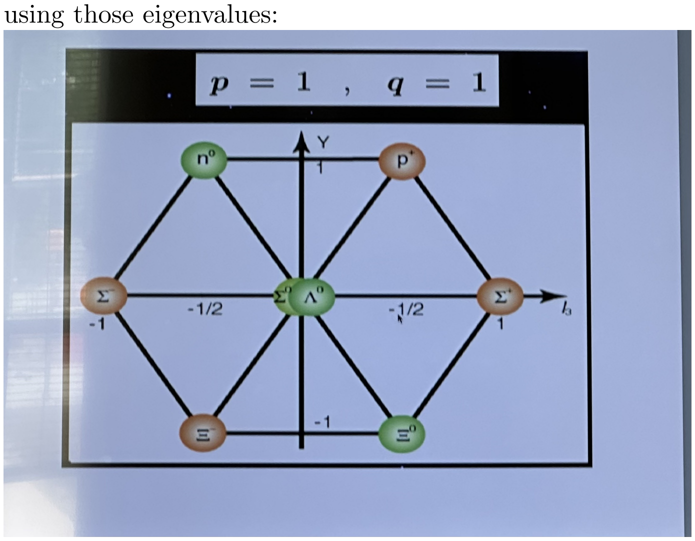
$$|-\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|0, -\frac{1}{\sqrt{3}}\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Where the labels in the ket are the eigenvalues in the two matrices respectively. We name things in order of how "close" they are to $SU(2)$, based on blocks of the matrices. We then plot the "names" of the eigenvectors in 2d space. This is not the space of eigenvectors, but rather the space of the **names** of the eigenvectors. We call this the **weight space**.

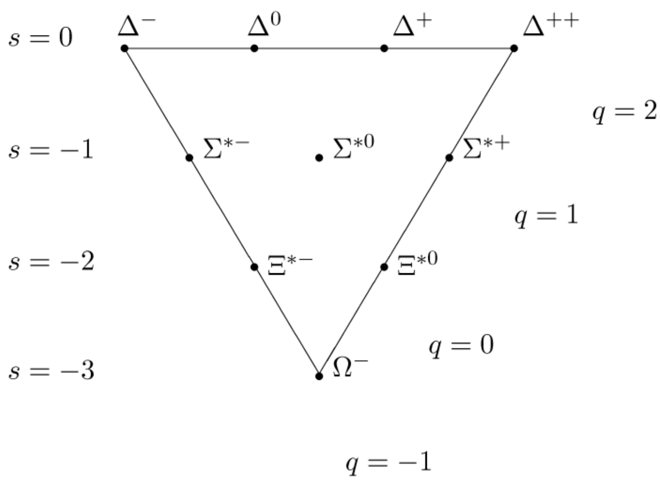
Recall the conjugate representation: we multiply generators by a minus sign, and conjugate them. The diagonal matrices are all real, so the conjugate representations of them are the same, but all entries negated. So all of the eigenvalues flip sign, corresponding to everything being reflected in the weight space. These two graphs in weight space correspond to the quarks and antiquarks

Now consider the 8×8 matrices in the adjoint representation, and find two that commute (they exist). Then we find their 8 eigenvalues, and plot their names using those eigenvalues:



We have all of these silly letters on each point, each corresponding to a particle. The interactions between fundamental particles are classified by the weight space. We have three axes along which particles have the same charge. There is also a property called strangeness that is also related to charge. The other diagonal orders particles according to strangeness. **Particles are representations of some algebra.**

The thing that really convinced the physics community is this:



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5.2 Back to the rotation group

Ladder operators:

$$(L_1 + iL_2)z = -iy + i(ix) = -(x + iy)$$

$$(L_1 + iL_2)(-(x + iy)) = -L_1(iy) - iL_2x = -iL_1y - iL_1x = -i(iz) - i(-iz) = 0$$

$$(L_1 - iL_2)z = x - iy$$

$$(L_1 - iL_2)(x - iy) = \cdots = 0$$

$$L_3z = 0$$

$$L_3(-(x + iy) = -(x + iy)$$

$$L_3(x - iy) = -(x - iy)$$

Note that these last three equations are eigenvalue equations, with eigenvalues of $1, 0, -1$. So we can call them:

$$|0\rangle = z$$

$$|1\rangle = -(x + iy)$$

$$|-1\rangle = (x - iy)$$

Now we can plot in the weight space, where the eigenvalues are coordinates. We end up just getting three points in a line. We have:

$$L_+|1\rangle = 0$$

$$L_-|-1\rangle = 0$$

$$L_+|-1\rangle = |1\rangle$$

$$L_+|-1\rangle = |0\rangle$$

$$L_+L_+|0\rangle = |1\rangle$$

We can notice that on the 8 state diagram, there are nodes similarly in a line. We can construct similar operators for $SU(3)$, based on the adjoint representation of the λs:

$$F_1 + iF_2$$

We can consider this similarly for the Pauli matrices, and get a weight space with only two points on the line (corresponding to $\pm\frac{1}{2}$).

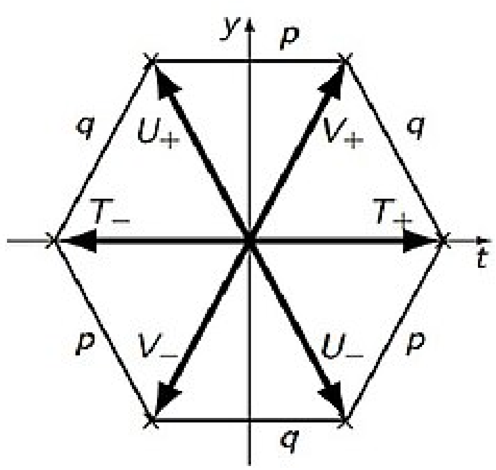
So we know that in the diagram for $SU(3)$ there are operators that move us along the horizontal axis. But what about the diagonal directions? We find that we can construct other operators that do this:

$$T_\pm = F_1 \pm iF_2$$

$$U_\pm = F_4 \pm iF_5$$

$$V_\pm = F_6 \pm iF_7$$

We can think of these as moving through a crystal:



When we "leave" our representation, things go to zero. We can however group our representation to be as large as we want. The size of our representation for $SU(3)$ (dimension of generator matrices) is given by Weyl's dimension formula:

$$d_{SU(3)} = \frac{1}{2}(p+1)(q+1)(p+q+2)$$

Each representation has the same number of matrices, but larger representations consist of larger matrices. So the matrices have more eigenvalues and thus more points in the weight space. Note that the number of commuting matrices is a property of the algebra. In $SU(3)$ there will be exactly 2 such matrices in any representation. We can also construct additional matrices that generate $SU(n)$ for $n > 3$, which is an entirely different process. Weight spaces are very commonly studied in mathematics, but not so much in physics.

Homework Problems:

1. Symplectic group:

$$Sp(2n, \star) = \left\{ A = \mathrm{Mat}(2n, \star) \mid A^T J A = J \ , \ J = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{pmatrix} \right\}$$

   Where $\star$ is any field. Compute:

$$\dim Sp(2n, \mathbb{R})$$

$$\dim 2p(2n, \mathbb{C})$$

2. USp group:

$$\begin{aligned} USp(2n, \mathbb{C}) &= \{A \in Sp(2n, \mathbb{C}) \mid A \in U(n,n)\} \\ &= \left\{ A \in Sp(2n, \mathbb{C}) \mid A^\dagger H A = H \ , \ H = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \right\} \end{aligned}$$

   Compute:

$$\dim USp(2n, \mathbb{C})$$

A representation is:

$$\rho : G \to \mathrm{End}_{\mathbb{R}}(V)$$

That is, a map from a group G to the endomorphisms of a vector space V. An endomorphism is a linear map from a vector space to the same vector space. So the endomorphisms we care about in general are matrices. We generally care about a "faithful" representation, where ρ is bijective.

Now consider a 2 sphere in 3 dimensions. The act of rotation simply moves a point on a sphere to another point on the sphere. So the points on the sphere form a vector space, and this space is the carrier space for the representation of rotations in 3 dimensions, $SO(3)$. Let n, p be points on the sphere, with $Rn = p$

where R is a linear transformation (i.e. matrix). Assume n is on the "north pole" of the sphere. Then it is also true that:

$$RR_3n = p$$

Since n is fixed by rotations about the z axis. In fact there are infinitely many possible ways to rotate and go from n to p. So:

$$p \sim \text{Rotations modulo } R_3$$

We call this the quotient group:

$$S_2 \cong SO(3)/SO(2)$$

Homework Problem: Find the parametrization of p in terms of u, v, coordinates introduced in stereographic projection, as a function of x, y, z, and vice versa.

Homework Problem: Find a new representation of rotation such that:

$$Ln = p$$

And satisfying:

$$L^T L = I$$

5.3 Special Relativity

We know that 2d rotations are given by:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

And so:

$$R(\theta)\vec{x} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\sin\theta \end{pmatrix}$$

And recall that the quantity $x^2 + y^2$ is invariant under this transformation:

$$|\vec{x}|^2 = |R(\theta)\vec{x}|^2$$

We can call this the Euclidean invariant. But what if we start with something else we want to keep invariant? We can call this the **Minkowski invariant**, and define it as:

$$I_2 = (ct)^2 - (x^2)$$

Suppose we say:

$$ct' = ct\cosh\Phi + x\sinh\Phi$$

$$x' = ct\sinh\Phi + x\cosh\Phi$$

Recall the definitions of hyperbolic trig functions:

$$\cosh \Phi = \frac{1}{2}\left(e^{\Phi} + e^{-\Phi}\right)$$

$$\sinh \Phi = \frac{1}{2}\left(e^{\Phi} - e^{-\Phi}\right)$$

And so:

$$\cosh^2 \Phi = \frac{1}{4}\left(e^{2\Phi} + e^{-2\Phi} + 2\right)$$

$$\sinh^2 \Phi = \frac{1}{4}\left(e^{2\Phi} + e^{-2\Phi} - 2\right)$$

And then taking their difference gives:

$$\cosh^2 \Phi - \sinh^2 \Phi = 1$$

Which is reminiscent of our favorite Pythagorean trigonometric identity, but with a minus sign. Which seems to hint that the matrix that preserves the Minkowski invariant will involve these functions. Perhaps:

$$R(\Phi) = \begin{pmatrix} \cosh \Phi & \sinh \Phi \\ \sinh \Phi & \cosh \Phi \end{pmatrix}$$

We can explicitly check that this maintains the invariance.

Relativity: an attempt to describe the laws of physics from different viewpoints. Two goals:

1. Describe physics from different observers.
2. Find universal truths; i.e. find properties of physical systems that do not change based on observer or coordinate system.

In special relativity, we assume the speed of light is finite, whereas in Newtonian relativity it is taken as infinite.

Now suppose we have two observers, each with their own coordinate system and clock; x, y, z, t and x', y', z', t'. Now suppose observer 2 is moving with velocity v relative to observer 1. How can we go from one coordinate system to the other? Suppose their is some linear map from one coordinate system to the other. So the transformation has the form:

$$x'(x,t) = \alpha x + \beta t + c_1$$

$$t'(x,t) = \gamma x + \delta t + c_2$$

We can appropriately adjust such that $c_1 = c_2 = 0$. Our goal is to find these coefficients. First we want to describe the motion of the origin of frame 2 from the perspective of frame 2:

$$x = vt$$

From the perspective of the second observer, their origin is always at the same point:

$$x' = 0$$

So:

$$0 = \alpha v t + \beta t$$

$$\beta = -\alpha v$$

So we now have:

$$x'(x,t) = \alpha(x - vt)$$

$$t'(x,t) = \gamma x + \delta t$$

We can now do the opposite, looking at the origin for observer 1:

$$x = 0$$

$$x' = -vt'$$

Plugging into the original equations forces:

$$\alpha = \delta$$

And so:

$$x'(x,t) = \delta(x - vt)$$

$$t'(x,t) = \gamma x + \delta t$$

Newtonian relativity makes the assumption that the times must be the same, concluding that:

$$t' = t$$

And so:

$$\gamma = 0$$

$$\delta = 1$$

But we have no real reason (beyond our own intuition) to assume that $t = t'$. In 1905, Einstein declared a new law of nature: light moves at the same speed in all reference frames. This is obviously not compatible with Newtonian relativity. In frame 1, an object moving at c would be described as:

$$x = ct$$

And in frame 2, it would be similarly described as:

$$x' = ct'$$

Why? Because Einstein said so. We can then plug these relations into our general equations to get:

$$\gamma = -\frac{\delta v}{c^2}$$

And so plugging in gives:

$$t'(x,t) = \delta\left(t - \frac{v}{c^2}x\right)$$

And we will now refer to δ as γ for the sake of convention:

$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

And so we see that time is not universal between observers. We now just need to determine the value of γ. Because the speed of light is constant in all reference frames, we must have:

$$(ct)^2 - x^2 = (ct')^2 - x'^2$$

Solving for γ gives:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

And of course:

$$\gamma^2 - \frac{v^2\gamma^2}{c^2} = 1$$

And so we can say:

$$\beta = \frac{v}{c}$$

Such that:

$$\gamma = \cosh\Phi$$

$$\beta\gamma = \cosh\Phi$$

And the Lorentz transforms just become a matrix of hyperbolic trigonometric functions. We can find the generator by Taylor expanding:

$$L_x = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Where rotation around the x axis is given by:

$$R_x(\Phi) = e^{-i\Phi L_x}$$

Now we just need to expand this to include rotations about x and y.

Homework Problem: Check the commutator algebra of the Lorentz transform matrices (4d rotations but with cosh/sinh)

6 June 8

Homework Problem: Prove or disprove:

$$x_{(i}x_{j)} - \frac{2}{3}\delta_{ij}x_kx_k = \begin{pmatrix} 2x^2 - \frac{2}{3}|\vec{r}|^2 & xy & xz \\ yx & 2y^2 - \frac{2}{3}|\vec{r}|^2 & yz \\ zx & zy & 2z^2 - \frac{2}{3}|\vec{r}|^2 \end{pmatrix}$$

6.1 curvature in differential geometry

What does it mean for an object to be "curved" mathematically?

We can specify any point on a cylinder with 2 numbers: an angle $0 \leq \phi < 2\pi$, and a height $0 \leq z \leq H$. We can write a vector from the origin to a point $P = (\phi, z)$ as:

$$\vec{r}(\phi, z) = R\cos(\phi)\hat{x} + R\sin(\phi)\hat{y} + z\hat{z}$$

Similarly, we can specify a point on a cone with a point and angle, with vector:

$$\vec{r}(\psi, z) = \frac{R(H-z)}{H}\left(\cos(\phi)\hat{x} + \sin(\phi)\hat{y}\right) + z\hat{z}$$

And for the sphere we of course have:

$$\vec{r}(\theta, \phi) = R\sin\theta\cos\phi\hat{x} + R\sin\theta\sin\phi\hat{y} + R\cos\theta\hat{z}$$

We can then apply the total differential operator, which does the following:

$$d(\vec{r}) = \sum dx_i \frac{\partial \vec{r}}{\partial x_i}$$

We then have a vector valued $d\vec{r}$, and so we can calculate $d\vec{r} \cdot d\vec{r}$ to give a scalar in terms of the differentials. We then write this as:

$$d\vec{r} \cdot d\vec{r} = \begin{bmatrix} d\phi & dz \end{bmatrix} A \begin{bmatrix} d\phi \\ dz \end{bmatrix}$$

Where A is a 2×2 matrix. This matrix is called **the metric**, as is often discussed in general relativity. Gauss thought about curvature by considering a small bit of the surface. If an ant was walking on the surface, how would they know if it's curved? Doing these calculations starts with the metric. There are infinitely many ways in which I can specify a point as a function of two variables, and our goal is a general procedure to determine if such a surface is curved.

Now we need to make this into a tensor, since Albert Einstein was a freak. Recall from last week that we can write a vector as a tensor with one index. We can denote the points on any of our three surfaces as:

$$\begin{aligned} u^i &= (\phi, z) \text{ for a cone} \\ &= (\phi, z) \text{ for a cylinder} \\ &= (\theta, \phi) \text{ for a cone} \end{aligned}$$

Where $i \in \{1, 2\}$, denoting components. We can eventually denote:

$$d\vec{r} \cdot d\vec{r} = du^i g_{ij} du^j = du^j g_{ij} du^i = (ds)^2$$

Where g_{ij} is the metric, i.e. the matrix we defined above. Also note that the metric is always symmetric under the exchange of its indices:

$$g_{ij} = g_{ji}$$

Recall the $d\vec{r}$ is the **line element**, and so this inner product is the magnitude of the line element squared. And so the metric is just a tensor that gives us the square magnitude of the line element.

In general, we say that surfaces are just parameterizations of multiple variables. The "nice" surfaces are the ones that are differentiable everywhere or almost everywhere. For example, the cone is differentiable everywhere except for a single point at the very tip. The metric may be a constant, or it might depend on our variables. In the case that it does depend on our variables, we can do calculus with the metric. For example, Einstein's field equations are actually second order differential equations of the metric.

We can also write:

$$d\vec{r} = dx^{\underline{m}} e_{\underline{m}}{}^{\underline{a}} \hat{\ell}^{\underline{a}}$$

Where dx^m is the differential of the coordinates, $e_{\underline{m}}{}^{\underline{a}}$ is a tensor, and $\hat{\ell}^{\underline{a}}$ is a unit vector. We might need to be careful to scale our unit vectors by appropriate factors, which might be a pain (like for the cone). We call this e the **frame**. We can also have an inverse frame, which is a tensor such that:

$$A_{\underline{m}}{}^{\underline{a}} B_{\underline{a}}{}^{\underline{m}} = \delta_{am}$$

Homework Problem: Calculate metric, frame, inverse frame, anholonomy, and curvature:

1. Non-Orthogonal plane: $\hat{x}, \hat{y} \to \hat{u}, \hat{v} \,|\, \cos(\hat{u}, \hat{v}) = \varphi_0$ Also consider:

$$\vec{r}(u, v) = u\hat{x} + v(\hat{y} + \cos(\varphi_0)\hat{x})$$

2. Torus

We now define the **Frame Operator**:

$$e_{\underline{a}} \equiv e_{\underline{a}}{}^{\underline{m}} \partial_{\underline{m}}$$

"And now — watch the magic" —Jim
Now let $u = u^i$, and so:

$$e_{\underline{a}} f(u) \text{ is calculable given } f(u)$$

And so we can also calculate:

$$(e_{\underline{a}} e_{\underline{b}} - e_{\underline{b}} e_{\underline{a}}) f(u) = C_{\underline{ab}}{}^{\underline{c}} e_{\underline{c}} f(u)$$

Note that we are in two dimensions, and so a, b take on values $1, 2$. So in 2d we have:

$$(e_{\underline{a}} e_{\underline{b}} - e_{\underline{b}} e_{\underline{a}}) f(u) = C_{\underline{ab}}{}^{\underline{c}} e_{\underline{c}} f(u) = \epsilon_{\underline{ab}} c^{\underline{c}} e_{\underline{c}} f(u)$$

Now we define an object:

$$\mathscr{R} = \epsilon^{\underline{ab}} [e_{\underline{a}} c_{\underline{b}} - e_{\underline{b}} c_{\underline{a}}] - k_0 c^a c_a$$

This is the **Gaussian mean curvature**. This is one of the simplest definitions we have for intrinsic curvature of a surface. If $\mathscr{R} = 0$, then the surface is not curved.

Homework problem: For the sphere, there is constant positive curvature. Do the same for the hyperbolic case:

$$x^2 + y^2 - z^2 = r^2$$

Or:

$$\vec{r}(u, v) = u\hat{x} + v\hat{y} + (u^2 - v^2)\hat{z}$$

Homework problem: Write the algebra of 4d rotations, and of the Lorentz group.

7 June 9

7.1 Notes on the homework

Recall that in our study of differential geometry we found a relationship between frame operators that looks suspiciously similar to the commutator relation of a Lie algebra:

$$[e_a, e_b] = C_{ab}{}^c e_c$$

That is, the commutator of the frame operators is a linear combination of frame operators. However this is **not** a Lie algebra, since the Cs are not necessarily constants, but can in fact be functions of our variables.

Note: when finding $\hat{\ell}_i$, all unit vectors must be **orthonormal** and tangent to the plane.

Another example:

$$\vec{r} = u\hat{x} + v(\hat{y} + \cos\phi_0 \hat{x})$$

$$d\vec{r} = du\hat{x} + dv(\hat{y} + \cos\phi_0 \hat{x})$$

$$d\vec{r} \cdot d\vec{r} = (dv)^2 + (du + \cos\phi_0)^2$$

And so:

$$g = \begin{bmatrix} 1 & \cos\phi_0 \\ \cos\phi_0 & 1 + \cos^2\phi_0 \end{bmatrix}$$

We can also write:

$$d\vec{r} = \begin{bmatrix} du & dv \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \cos\phi_0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

Which gives inverse frame:

$$e = \begin{bmatrix} 1 & 0 \\ -\cos\phi_0 & 1 \end{bmatrix}$$

Hence we have the frame operators:

$$e_1 = \frac{\partial}{\partial u}$$
$$e_2 = -\cos\phi_0 \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

And these commute, so there is zero curvature.

Now consider the Torus, centered at the origin, with the donut oriented in the xy plane. We can consider two angles:

1. ϕ which gives the angle in the xy plane.

2. θ which gives the angle from the xy plane.

Also suppose we have two radii, r_0 being the radius of the cross section, and R_0 the radius of the center of the donut. We can parametrize this as:

$$\vec{r} = (R_0 + r_0\cos\theta)(\cos\phi\hat{x} + \sin\phi\hat{y}) + r_0\sin\theta\hat{z}$$

This gives the differential:

$$d\vec{r} = r_0(-\sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) + \cos\theta\hat{z})d\theta + (R_0 + r_0\cos\theta)(-\sin\phi\hat{x} + \cos\phi\hat{y})d\phi$$

And so we can split this into our frame, and find the inverse frame and corresponding frame operators. Remember to check that our unit vectors are orthogonal! We then look at frame operators etc. to find curvature.

Note: "Math becomes context dependent, just like language. If I am standing on the beach, I might say "there's a sea there". If I'm standing at a blackboard I might say "I have a 'C' here." — Jim

Some more Special Relativity:
The Lorentz group is just the group that preserves:

$$I = -(ct)^2 + x^2 + y^2 + z^2$$

With corresponding matrix:

$$\eta_{mn} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We often denote it as $SO(1,3)$. We can similarly denote a group $SO(p,q)$, in which the invariant quantity is:

$$I = -x_1^2 - x_2^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{p+q}^2$$

Which we can think of as a system having p many "time dimensions" and q many spacial dimensions. There is a corresponding matrix that is like the one above.

7.2 Konstantinos teaches us witchcraft mathematics

Gaussian curvature in the way we calculated it is not really intrinsic, since we have only considered a surface embedded in $\mathbb{R}^3$. The true intrinsic curvature can only depend on the space itself, and not some embedding.

"You see you have been living in a dream, and now I have to fill your mind. You are living in a matrix, matrix is a human construction." — Konstantinos

Consider a matrix. It is an array of numbers, and so it has two indices. Once for the row, and one for the column. For example:

$$A = \left(\right)_{ij}$$

Where i counts rows and j counts columns. These indices are distinct, and serve different purposes. We should not treat them the same way, but we often do. In order to convey that they are distinct types of indices it to change notation a little bit:

$$A = \left(\right)_{i}{}^{j}$$

$$A = \left(\right)^{i}{}_{j}$$

We typically write the first index 'down', to denote the rows, and the second 'up', denoting columns. Remember: left and right matter:

$$A_i^j \neq A_i{}^j \neq A^j{}_i$$

A vector has only rows, and thus only needs a single index. We typically right that index 'down'. Similarly a row vector has only columns, and thus only a single index 'up'. For example:

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix}$$

$$v = \begin{pmatrix} v^1 & v^2 & \dots & v^n \end{pmatrix}$$

Transposition is not as simple as changing indices, despite what we may hope.

A field K is a set together with two binary operations, $+$ and $\cdot$:

$$+ : K \times K \to K$$

$$\cdot : K \times K \to K$$

They have the "CANI" properties: commutativity, associativity, neutral element (identity), inverses (except the annoying issue of zero).

We now take a little bit of a detour to define rings. A set R equipped with a similar $+, \cdot$. Where $+$ obeys the CANI properties, but multiplication is associative only, without guaranteed inverses or commutativity. We (usually) assume rings contain unity.

Now we talk about a K vector space, where K is an arbitrary field. A vector space is a set V, equipped with with two operations:

$$\oplus : V \times V \to V$$

$$\odot : K \times V \to V$$

Such that $\oplus$ satisfies CANI, and $\odot$ satisfies "ADDU": associative, distributive over $\oplus$, distributive over $+$, and unital.

Perhaps the most obvious example is position vectors, also known as "configuration space". We have a coordinate system with each point described as:

$$P = (x, y, z)$$

Where we call P a "vector". Another example: $m \times m$ matrices over $\mathbb{R}$. A third example: functions $f(x)$ defined on the interval $[0, 1]$.

Now consider a map between vector spaces:

$$f : V \to W$$

In order to preserve structure, our map must be linear:

$$f : V \tilde{\to} W$$

Meaning it satisfies:

$$f(V_1 \oplus V_2) = f(V_1) \oplus f(V_2)$$
$$f(\lambda \cdot V) = \lambda \cdot f(V)$$

Note that both vector spaces must be over the same field (or one field can contain the other). Now we can talk about:

$$\mathrm{Hom}(V, W) = \{f : V \tilde{\to} W\}$$

Which in turn forms a vector space under the operations:

$$(f + g)(v) = f(v) + f(g)$$

$$\lambda \cdot g(v) = \lambda \cdot g(v)$$

Now we can talk about:

$$\mathrm{End}_K(V) = \mathrm{Hom}(V, V)$$

So the space of endomorphisms is just the vector space of operators on V. We can also talk about:

$$\mathrm{Aut}_K(V) = \{\mathrm{End}_K(V) \mid \text{that are invertible}\}$$

Now we can discuss the following:

$$V^\star = \mathrm{Hom}(V, K)$$

Id est, the space of linear maps from V to the underlying field K. Or in other words, maps that "eat" a vector and spit out a number. The existence of tensors really just ties back to linear algebra.

Now we introduce the idea of a tensor. A p, q tensor over a field K is a multilinear map:

$$T : \underbrace{V \times V \times \cdots \times V}_{p \text{ many}} \times \underbrace{V^\star \times V^\star \times \cdots \times V^\star}_{q \text{ many}} \to K$$

Meaning:

$$T(v_1, v_2, \ldots, v_p, w_1, w_2, \ldots w_q) \in K$$

And the map is linear in each slot. We can say:

$$\begin{aligned} T_q^p V &= \underbrace{V^\star \otimes V^\star \otimes \cdots \otimes V^\star}_{p \text{ times}} \otimes \underbrace{V \otimes V \otimes \cdots \otimes V}_{q \text{ times}} \\ &= \left\{ T : \underbrace{V \times V \times \cdots \times V}_{p \text{ times}} \times \underbrace{V^\star \times V^\star \times \cdots \times V^\star}_{q \text{ times}} \tilde{\to} K \right\} \end{aligned}$$

We say that the rank of the tensor is $p + q$, and p is the valence of the tensor. So a tensor takes in p elements of V and q elements of $V^\star$. We can consider tensors as a vector space. We write:

$$(T_1^1 + S_1^1)(v, w) = T_1^1(v, w) + S_1^1(v, w)$$

For $v \in V$ and $w \in v^\star$. Also:

$$(\lambda \cdot T_1^1)(v, w) = \lambda \cdot T_1^1(v, w)$$

Now for some examples:

1. $T_0^0 V = K$ are the scalars.
2. $T_0^1 V = V^\star$ which we may call the "covectors" or "dual vectors".
3. $T_1^1 V = \mathrm{End}(V^\star)$ or in other words, matrices.
4. $T_1^0 V = V$ is our initial vectors.

5. $T^2_0 V = g$ is the metric. The metric is the object such that:

$$g(v_1, v_2) \in K$$

An interesting note:

$$\delta_{ij} \neq \delta_i{}^j$$

While they often act similarly, δ_{ij} is a metric we call the Kronecker delta, but $\delta_i{}^j$ is the identity matrix. The most common confusion amongst "physics people and sometimes mathematicians" is the difference between a metric and a matrix.

Now time to talk about bases of a vector space. A (Hamel) basis B is a finite subset of a vector space V such that:

$$B = \{e_1, e_2, \ldots, e_n\}$$

$$\sum_1^n \lambda^i e_i = 0 \;\Rightarrow\; \lambda^i = 0$$

$$\forall v \in V \;,\; \exists! \lambda^i \in K \; s.t. \; v = \sum_1^n \lambda^i e_i$$

Note that we cannot write these sums if we have an uncountable infinite basis. In the case of a countable infinite basis, we can write the sum but we then have to deal with the issue of convergence.

An example: consider u^i, $i = 1, \ldots, d$, What is a basis for the vector space of differential operators?

$$B = \left\{ \frac{\partial}{\partial u^i} \right\}$$

Are they linearly independent? Consider:

$$\lambda^i \frac{\partial}{\partial u^i} = 0$$

Test for the specific function:

$$f(u) = u^j$$

$$\lambda^i \frac{\partial}{\partial u^i}(u^j) = \lambda^i \delta_i{}^j = 0$$

$$\lambda^j = 0$$

So B is a linearly independent set. So to show that they are a basis we need to show that they span the space (we will come back to this).

We now have the tools to understand:

$$e_a = e_a{}^m \partial_m$$

$$e_a = {e_a}^i \frac{\partial}{\partial u^i}$$

Means that our differential operator can be expanded in the basis given above, with components ${e_a}^i$s. So e_a is the expansion of our vector in the B basis above. Similarly:

$$[e_a, e_b] = {C_{ab}}^c e_c$$

Is also an element of the same vector spaces, with coordinates ${C_{ab}}^c$ in the B basis. However note that partial derivatives commute, i.e.:

$$[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] = 0$$

We call this the u^i coordinates **holonomic basis**. We can act with the inverse:

$${e_j}^a e_a = \frac{\partial}{\partial u^j}$$

In order to "properly" do differential geometry, we need to talk about differential operators as vectors like we did here.

Let V be a vector space with some basis $B = \{e_1, e_2, \ldots, e_d\}$. Then we can write any vector $v \in V$ as:

$$v = v^i e_i$$

Note that by this convention the 'up' index on v corresponds to a row vector (an object with only columns). Then we also have another vector space $V^\star$, with a basis:

$$B_{V^\star} = \{\epsilon^2, \epsilon^2, \ldots, \epsilon^d\}$$

And we can express some $w \in V^\star$ as:

$$w = \epsilon^i w_i$$

And again, note that the notation here implies that w is a column vector. Now we can pick a simple tensor, for example:

$$\phi \in T^1_1 V$$

So ϕ takes one vector and one covector, and outputs a scalar. So we can consider $\phi(v, w)$:

$$\begin{aligned}
\phi(v, w) &= \phi\left(\sum v^i e_i, \sum \epsilon^j w_j\right) \\
&= \sum\sum \phi(v^i e^i, \epsilon^j w^j) \\
&= \sum\sum v^i w_j \phi(e_i, \epsilon^j)
\end{aligned}$$

Now let:

$$\phi(e_i, \epsilon^j) = {\phi_i}^j$$

And so:

$$\phi(v, w) = \sum_i \sum_j v^i \phi_i{}^j w_j$$

So we can interpret this as matrix multiplication, giving:

$$\phi(v, w) = v\phi w$$

Where we are matrix multiplying a row vector, a matrix, and a column vector. Note: if we use Einstein notation to omit the sums, we would be in *big* trouble if it weren't for the fact that all of our maps are linear. It is nice for tensors which are multi-linear, but be very careful elsewhere.

Now we consider the example of a tensor of the form:

$$w \in T_0^1 V$$

So it 'eats' one vector and no covectors, and hence is a covector. We can similarly expand it in our given basis:

$$w = \epsilon^j w_j$$

And so:

$$\begin{aligned} w(v) &= \epsilon^j(v) w_j \\ &= \epsilon^j(v^i e_i) w_j \\ &= v^i \epsilon^j(e_i) w_j \\ &= v^i C_i{}^j w_j \end{aligned}$$

Which we can think of as a row vector v multiplied by a matrix C and a column vector w. What if we demand that:

$$C_i{}^j = \delta_i{}^j$$

Then we get:

$$v \cdot w$$

which is the traditional definition of an inner product (with appropriate transpositions and conventions. When we have:

$$\epsilon^j(e_i) = \delta_i{}^j$$

We call it the **dual basis**.

Now how can we construct new tensors from old tensors? One way is called the **tensor product**. Consider two tensors: T_q^p and S_s^r. We can construct a new tensor denoted as:

$$T_q^p \otimes S_s^r = A_{q+s}^{p+r}$$

We have:

$$\begin{aligned}
&(T^p_q \otimes S^r_s)(v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+r}, w_1, \ldots, w_q, w_{q+1}, \ldots, w_{q+s}) \\
&= T^p_q(v_1, \ldots, v_p, w_1, \ldots w_q) \cdot S^r_s(v_{p+1}, \ldots, v_{p+r}, w_{q+1}, \ldots, w_{q+s})
\end{aligned}$$

Homework problem: figure out the tensor product of 2×2 matrices.

So consider a tensor T^1_1, with $T(e_i, \epsilon^j) = T_i{}^j$. Now consider:

$$T = T_i{}^j e_j \otimes \epsilon^i$$

So:

$$\begin{aligned}
T(e_k, \epsilon^l) &= T_i{}^j e_j \otimes \epsilon^i(e_k, \epsilon^l) \\
&= T_i{}^j e_j(\epsilon^l) \cdot \epsilon^i(e_k) \\
&= T_k{}^l
\end{aligned}$$

What happens when we change basis? In physics, we generally do calculations with respect to an agreed upon basis. In the math world, it is generally frowned upon to choose a basis, and we instead try to do everything abstractly.

When we transform a vector, there are two things we can do. We can leave the coordinates the same, and move the vector, or we can think of the vector as being unchanged, and rotate the coordinate system around it. The first case is called an **active transformation**, and the second is a **passive transformation**. For now we will focus on a change of basis in a passive transformation.

Consider a vector $v \in V$:

$$v = v^i e_i$$

And now consider some map:

$$\{e_i\} \to \{\tilde{e}_i\}$$

$$e_i = T_i{}^j \tilde{e}_j$$

So we can write the same vector with respect to the new basis:

$$v = \tilde{v}^i \tilde{e}_i$$

We can find the components of this new vector:

$$v = v^i T_i{}^j \tilde{e}_j$$

$$\tilde{v}^i = v^i T_j{}^i$$

We can do the same thing in the ϵ basis for a covector:

$$w = \epsilon^i w_i$$

$$\epsilon^i \to \tilde{\epsilon}^i$$

With:

$$\epsilon^i = \tilde{\epsilon}^j B_j{}^i$$

So:

$$\tilde{w}_i = B_i{}^j w_j$$

So the moral of the story is that under coordinate changes, the vector components transform with the transformation matrix from the right, and the covectors from the matrix to the left. We have:

$$\tilde{w}_i = B_i{}^j w_j$$

$$\tilde{v}^i = v^i A_j$$

So it follows that A and B are inverses of each other (however this relies on us using the dual basis).

Now what does this mean for tensors? Suppose we have components of a pq tensor:

$$T^{j_1,\ldots,j_q}_{i_1,\ldots,i_p} = T(e_{i_1},\ldots,e_{i_p},\epsilon^{j_1},\ldots,\epsilon^{j_q})$$

How would these components change under a coordinate transform? It would look something like:

$$\tilde{T}^{j_1,\ldots j_q}_{i_1,\ldots,i_p} = (A^{-1})_{i_1}{}^{k_1}(A^{-1})_{i_2}{}^{k_2}\ldots(A^{-1})_{i_p}{}^{k_p}T_{k_1,\ldots,k_p}{}^{l_1,\ldots,l_q}(A)_{l_1}{}^{j_1}(A)_{l_2}{}^{j_2}\ldots(A)_{l_q}{}^{j_q}$$

Now we want to emphasize the difference between a matrix and a metric. Consider a matrix ϕ, i.e. a T^1_1 tensor. Then under a coordinate transformation:

$$\tilde{\phi} = A^{-1}\phi A$$

A metric g, i.e. a T^2_0 tensor transforms as:

$$\tilde{g} = A^{-1}A^{-1}g$$

Which is a little different. Another interesting note: determinant of a matrix is invariant under change of basis. The same is not true for the metric.

Homework problem: Consider a metric g (a tensor that takes 2 vectors and gives a scalar). Consider the matrix such that:

$$g(Mv_1, v_2) = g(v_1, M^T v_2)$$

Find an explicit representation for M^T:

$$(M^T)_i{}^j = ?$$

8 June 10

8.1 Notes on the homework

Tensor product of matrices:

$$T_1^1 \otimes S_1^1 = A_{1+1}^{1+1} = A_2^2$$

So the result is a map that takes 2 vectors and 2 covectors to a scalar:

$$\begin{aligned} T_1^1 \otimes S_1^1(v_1, v_2, w_1, w_2) &= T_1^1(v_1, w_1) \cdot S_1^1(v_2, w_2) \\ &= {v_1}^i {T_i}^j w_{1j} \cdot {v_2}^k {S_k}^l w_{2l} \\ &= {v_1}^i {v_2}^k {T_i}^j {S_k}^l w_{ij} w_{2l} \end{aligned}$$

And so we can write:

$${T_i}^j {S_k}^l = A_{(i,k)}^{(j,l)}$$

Id est, we have a 4×4 matrix:

$$A = \begin{pmatrix} {T_1}^1 S & {T_1}^2 S \\ {T_2}^1 S & {T_2}^2 S \end{pmatrix}$$

We can generalize this in arbitrary dimensions:

$$A_{m \times m} \otimes B_{n \times n} = C_{mn \times mn}$$

Now it's time for transposes. The following is the definition of transpose:

$$g(Mv, u) = g(v, M^T u)$$

The property of having a transpose requires a metric. Note that $v, w \in V$. With indices we can write:

$$g(Mv, u) = g({M_k}^i v^k e_i, u^j e_j) = {M_k}^i v^k u^j g(e_i, e_j)$$

And for the right hand side:

$$g(v, M^T u) = v^k u^j {(M^T)_j}^i g_{ki}$$

And so:

$${M_k}^i g_{ij} = {(M^T)_j}^i g_{ki}$$

We can also define an object g^{ij}, such that:

$$g_{ki} g^{ij} = {\delta_k}^i$$

We then rewrite:

$$g^{kl} {M_k}^i g_{ij} = {(M^T)_j}^i g_{ki} g^{kl}$$

And so:

$${M_k}^i g^{kl} g_{ij} = {(M^T)_j}^l$$

So in order to have a well defined transpose, we need a metric, which we can think of as determining the inner product (since an inner product maps 2 vectors to a scalar). So this is really just equivalent to the statement we have seen before:

$$\langle Mv, u\rangle = \langle v, M^T u\rangle$$

An observation: We can take a matrix v^i and a metric g_{ij} to define a covector:

$$v^i g_{ij} = w_j$$

Which way may denote as v_j, to denote that it is the covector that corresponds to the vector v^i. So we can use a metric to bring indices down. Similarly, we can use an inverse metric g^{ki} and a covector w_i to define:

$$g^{ki} w_i = w^k$$

And hence we can use an inverse metric to convert a covector to the corresponding vector. Using we can rewrite out equation for transpose from above:

$$M_{kj} = (M^T)_{jk}$$

But note that these objects are not matrices anymore, but rather $2, 0$ tensors.

"Most people don't see this stuff until graduate school. Lots of people think it will be like the icing on the cake. Then you realize you're actually in the basement and you have to dig your way to the top." — Jim

8.2 Preparing to meet Uncle Albert

Non orthogonal coordinates in the 2d plane. We can write a position vector as:

$$\vec{r}(u, v) = u\hat{x} + v(\hat{x}\cos\phi_0 + \hat{y}\sin\phi_0)$$

And note that ϕ_0 cannot be $0, \pi$ or else our coordinate system doesn't work. We also then have:

$$d\vec{r}(u, v) = du\hat{x} + dv(\hat{x}\cos\phi_0 + \hat{y}\sin\phi_0)$$

Also note that we now have two unit vectors to start with (they don't have to be orthogonal but they do need to be linearly independent). Now that we have the unit vectors we can write the gradient:

$$\vec{\nabla} = \hat{x}e_u + (\hat{x}\cos\phi_0 + \hat{y}\sin\phi_0)e_v$$

That is, the unit vectors times some frame operators that we don't know yet. It is *always* true that:

$$d = d\vec{r}\cdot\vec{\nabla}$$

We know that e_u, e_v are differential operators. De can then write the dot product:

$$d\vec{r}\cdot\vec{\nabla} = du[e_u + \cos\phi_0 e_v] + dv[\cos\phi_0 e_u + e_v]$$

This forces the conditions:

$$e_u + \cos\phi_0 e_v = \frac{\partial}{\partial u}$$

$$\cos\phi_0 e_u + e_v = \frac{\partial}{\partial v}$$

And so we can solve these for e_u, e_v:

$$e_u = \frac{1}{\sin^2\phi_0}\left[\frac{\partial}{\partial u} - \cos\phi_0\frac{\partial}{\partial v}\right]$$

$$e_v = \frac{1}{\sin^2\phi_0}\left[-\cos\phi_0\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right]$$

And so we can get a tensor explicitly from this. We did this to explicitly derive similar results for spherical and cylindrical coordinates.

We successfully discussed Gaussian mean curvature in 2 dimensions, but Uncle Albert requires us to work in 4 dimensions (3 spacial and 1 time). So we need to work our way up to that, starting with 3d. The key takeaway is that the d operator is the same in any coordinate system. So we have a symmetry.

8.3 Beginning our journey into Uncle Albert's playground

"Are we ready to begin our journey into Uncle Albert's playground?" — Jim

There is a symmetry principle that is related to general relativity: the differential operator.

Consider the position vectors for a sphere:

$$\vec{r} = r\cos\phi\sin\theta\hat{x} + r\sin\phi\sin\theta\hat{y} + r\cos\theta\hat{z}$$

How can we generalize this into a parametrization for a 3 sphere? Using inspiration from the above:

$$\vec{r}_4 = r\{[\cos\phi\hat{x} + \sin\phi\hat{y}]\sin\beta + [\cos\theta\hat{z} + \sin\theta\hat{q}]\cos\beta\}$$

Now we can dot it with itself and see what happens:

$$\vec{r}_4 \cdot \vec{r}_4 = r^2$$

We call this the **Hopf coordinate description** of a 4d sphere. We can also write an alternative description that is also somewhat natural:

$$\vec{r}_4 = r[\cos\phi\sin\theta\hat{x} + \sin\phi\sin\theta\hat{y} + \cos\theta\hat{z})\sin\beta + \cos\beta\hat{q}]$$

This is the more standard way. This is called **hyper spherical coordinates**, and can be naturally extended to higher dimensions. We do so by multiplying the previous parametrization by sin of a new angle, and adding a unit vector multiplied by cos, and maintaining an overall factor of r. Hopf coordinates are only well defined in even dimensions.

Just like in 3d we can take a total differential, guess the e operators, and then use that to calculate the anholonomy.

Suppose we do this with the sphere in $4d$. We can consider the hyper spherical coordinates with constant radius:

$$\vec{R}(\phi, \theta, \beta)$$

And then we can calculate the operators as above:

$$e_\phi = \frac{1}{r \sin\beta \sin\theta} \frac{\partial}{\partial\phi}$$

$$e_\theta = \frac{1}{r \sin\beta} \frac{\partial}{\partial\theta}$$

$$e_\beta = \frac{1}{r} \frac{\partial}{\partial\beta}$$

And from there we can get commutators:

$$[e_\phi, e_\theta] = \frac{\cos\beta}{r \sin\beta \sin\theta} e_\phi$$

$$[e_\theta, e_\beta] = \frac{\cos\beta}{r \sin\beta} e_\theta$$

$$[e_\beta, e_\phi] = -\frac{\cos\beta}{r \sin\beta} e_\phi$$

From here we can calculate the Riemann curvature tensor:

$$\mathcal{R}_{abcd} = \{e_a w_{bcd} - e_b w_{acd} - c_{ab}{}^k w_{kcd} - w_{ac}{}^k w_{bkd} + w_{ad}{}^k w_{bkc}\}$$

Where:

$$w_{abc} = w_{abc}(e) \equiv \frac{1}{2}[c_{abc} - c_{acb} - c_{bca}]$$

$$c_{ab}{}^c \equiv [e_a{}^m \partial_m e_b{}^n - e_b{}^m \partial_m e_a{}^n] e_n{}^c$$

Recall that our indices can take on three values corresponding to our three coordinates, ϕ, θ, β. A note on notation: c_{abc} means the coefficient corresponding to c of the ab commutator. Also note that using symmetry arguments we can significantly reduce the number of terms we need to calculate in the Riemann curvature tensor.

9 June 11

9.1 Homework

Konstantinos: "Guys this is not good. I'm not talking about the results. You know what I'm talking about."

We recall that the curvature of a 2 sphere is constant and positive. We would realistically expect the same thing for the 3 sphere.

Homework problem: Compute the curvature of the 3 sphere (as a group).

9.2 Brief review

Using our convention, we specify a vector using its components, with an index on the top:

$$v = v^i$$

We specify a matrix with two indices, the first one for rows and one for columns:

$$A = {a_i}^j$$

And so this notation means v^i has only columns, and is thus a row vector. Now recall our notation for anti-symmetrization:

$$A_{[ab]} = A_{ab} - A_{ba}$$

However in some other textbooks:

$$A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})$$

Or similarly:

$$A_{[a_1 \ldots a_n]} = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) A_{\sigma(a_1) \ldots \sigma(a_n)}$$

However we do not use that notation here. Note that our conventions require that a vector (row vector) be multiplied to the left of a matrix, and a (column) covector to the right.

We can characterize how a vector transforms under a linear transformation:

$$A : V \to V$$

$$v^i A = v^j {A_j}^i$$

And similarly we can transform covectors:

$$A^{-1} \omega_i = {(A^{-1})_i}^j \omega_j$$

Recall that we are choosing the dual basis, such that:

$$\epsilon^i(e_j) = \delta_j{}^i$$

If we demand that the new basis after a transformation maintain this property, then we found that:

$$A_i{}^j = (B^{-1})_i{}^j$$

This imposes the condition that A must be invertible. Also note that the inner product does not depend on basis, so this definition states that:

$$v^i w_i = \tilde{v}^i \tilde{w}_i$$

Now consider a linear transformation in n dimensional Euclidean space. Suppose our linear transformation R is a rotation. So R can be represented as an $n \times n$ matrix:

$$R_1 R_2 : v^i \to \tilde{v}^i = v^j (R_j)^i \to \tilde{\tilde{v}} = \tilde{v}^j R_{2}{}_j^i = v^k (R_1 R_2)_k{}^i$$

However this is problematic, since composing the rotations would seem to give:

$$\tilde{\tilde{v}}^i = v^k (R_2 R_1)_k{}^i$$

So we need to actually apply the inverse of the rotation matrices:

$$\tilde{\tilde{v}}^i = v^k ((R_2 R_1)^{-1})_k{}^i$$

And so objects with an up index (our row vectors) must transform according to the inverse of a rotation element, not the element itself. Note that we could solve this problem similarly with transposition or Hermitian conjugation. We call this a "left representation", since we care about order from the left. This same process holds for any group of transformations (such that closure is satisfied and inverses exist).

Now consider a transformation of an object:

$$w_i \to \tilde{w}_i = R_i{}^j w_j$$

Which after a second transformation gives:

$$\tilde{\tilde{w}}_i = (R_2 R_1)_i{}^k w_k$$

Which does not cause issues, and the appropriate group structure is satisfied.

9.3 This business of intrinsic curvature

A core assumption of physics in relativity is that spacetime is a continuum. It is a stage on which all physics phenomena take place, with all events described by real numbers describing 4 coordinates. The dynamics of this space is determined by the distribution of energy in spacetime. Compare this to Newtonian physics, where space and time are completely constant regardless of any factors. Einstein made a radical statement: matter and energy directly effect what space and time are. In order to effectively do this, we need a device that allows us to characterize curved space time.

Gauss discovered that a surface has two different kinds of curvature. **Extrinsic curvature**, characterized by whether or not a surface deviates from a plane. For example, a cylinder has extrinsic curvature since it is not part of a plane. He also discussed **intrinsic curvature**. Imagine having a piece of paper we can freely deform, and on it we can write geometric figures: line segments are the shortest paths between two lines, circles are collections of points equidistant from a center, triangles with angles summing to 180°. Now imagine we take that paper and we bend it with those figures on it. All of our figures are now 'bent' as well, but the geometric relations still hold. The Pythagorean theorem is still true, the line is still the shortest path between two points, etc. All of the geometry on the "flat" paper still holds, and is in fact indistinguishable from that of the plane. We call this bent piece of paper "intrinsically flat", since it has extrinsic curvature, but the intrinsic geometry is the same.

Not all surfaces are intrinsically flat. For example, you can draw triangles on the surface of the sphere with angles that don't sum to 180° (in fact we can have a triangle with 3 right angles). This tells us that we are no longer doing Euclidean geometry, and in fact we must have some intrinsic curvature. Finding and characterizing this curvature is the purpose of differential geometry.

The past few days we have been taking an extrinsic approach, considering our curved surface as being embedded in some bigger space, and relying on the properties of that ambient space. This works, but has practical drawbacks since it requires extra dimensions. Our goal is to find a way to find the intrinsic curvature without referring to this ambient space. This will lead us down the deep dark path of manifolds and differential geometry. We will begin this path but won't make it too far.

There are many ways to go about this intrinsic description, but the strategy that often works is to find some strategy that works on a flat surface but not on a curved one, and go from there. The operation we start with is the idea of vector addition by "moving" one vector to the other. For example, consider a vector oriented tangent to a sphere pointing in some direction at the pole. We can move it down to the equator, shift it along the equator, then back to the pole, all without rotating the vector. The vector now points in a different direction. This is something we can do with no reference to the ambient space, for example by an ant moving along the surface.

"I am going to be very dramatic. From this point forward there is no ambient space. And if there is you are not allowed to look outside. You life on the surface" — Konstantinos

Suppose we have two points p, q on a curved surface M. If a particle passes through point p, we can describe its velocity by some vector tangent to the curve at p. We can call this T_pM, denoting a tangent vector at point p, on the surface M. Now suppose this particle passes through q with some other velocity, which we can similarly call T_qM. Both of these velocities are elements of the tangent space. Now suppose I use the ambient space to move T_pM from p to q in a parallel way like we did on the plane. But now the vector is no longer in the tangent space, and thus is no longer a valid velocity. Say $T_pM = v$ and our transported vector is $\tilde{v}$. We can however consider the projection of $\tilde{v}$ onto the tangent space, which we can call $\tilde{v}_P$. However this projection requires some additional information on how to do the projection, particularly if the tangent space is > 2 dimensions. We can have a mathematical object that does this projection (and gives us the required extra data), which we call a "connection". We now go back into the cage, and can only look at the intrinsic geometry.

"And at this point we go to a commercial break" — Konstantinos

9.4 Now we return from the commercial break: differential geometry from the intrinsic viewpoint

We are now forbidden from looking at the ambient space. Is every possible intrinsic geometry embeddable into an ambient space? This is unknown by mathematicians, and so it is definitely unknown by physicists.

Now we consider a manifold M, and a point p, with a particle moving through point p. Say this path is parametrized by some function $\gamma(t)$, which we can think of like time. We can say this object is at point p at time $\gamma(0)$. Then the directional derivative operator at p along the curve γ is defined by the linear map:

$$X_{\gamma,p} : C^\infty(M) \to \mathbb{R}$$

We can define functions on the manifold that map a point on the manifold to real numbers:

$$f : M \to \mathbb{R}$$

And we say: $f \in C(M)$ If f is continuous, and we say:

$$f \in C^\infty(M)$$

If f is infinitely differentiable. Note: in physics, we say that a manifold that has points, and is Hausdorff complete in patches. Now we return to our map $X_{\gamma,p}$, which returns the following number given an input function f:

$$f \mapsto X_{\gamma,p} f = [f(\gamma(\lambda))]'|_{\lambda=0}$$

How do we take the derivative? We have $\lambda \in \mathbb{R}$, and the map $\gamma(\lambda)$:

$$\gamma : \mathbb{R} \to M$$

And then:

$$f : M \to \mathbb{R}$$

And so:

$$f(\gamma(\lambda)) : \mathbb{R} \to \mathbb{R}$$

Meaning we are differentiating a function from the $\mathbb{R}$ to $\mathbb{R}$, which is easy. So:

$$[f(\gamma^a(t))]' = [f(x^1(t), x^2(t), \ldots, x^d(t))]' = \frac{\partial f}{\partial x^i}\frac{dx^i}{dt}$$

Where a is the vector components of the manifold according to the dimension of the manifold. And so:

$$[f(\gamma^a(\lambda))]'|_{\lambda=0} = \frac{\partial f}{\partial x^i}\frac{dx^i}{dt}\Big|_{\lambda=0} = \frac{\partial f}{\partial x^a}\Big|_{x^a=\gamma^a}(\lambda)\frac{d\gamma^a(\lambda)}{d\lambda}\Big|_{\lambda=0}$$

And so our directional derivative operator is:

$$X_{q,p} = v^m \frac{\partial}{\partial x^m}\Big|_p$$

Where:

$$v^a = \frac{d\gamma^a(\lambda)}{d\lambda}\Big|_{\lambda=0}$$

In this basis our operators commute. However we can consider another basis of differential operators: the e_as, which we say are different differential operators that span the same vector space:

$$X_{\gamma,p} = \tilde{v}^a e_a$$

Note that in this basis, our basis operators do not necessarily commute, and so we have the relation:

$$[e_a, e_b] = C_{ab}{}^c e_c$$

Now suppose that instead of having a single function f, we have many functions in an array, f^I. We can now introduce a new object:

$$\frac{\partial V^a(x)}{\partial x^M} = \lim_{dx\to 0}\frac{V^a(x+dx) - V^a(x)}{dx}$$

Where V is a vector depending on the coordinates of the manifold, x. That is:

$$V = V^a(x)e_a$$

We can now think of $V^a(x)$ and $V^a(x + dx)$ as two distinct points on the manifold. The tangent spaces (corresponding to the derivative at each point) are entirely separate. However this leads to an issue, since how can we subtract those vectors in completely separate spaces. In a flat plane, the tangent spaces are the same, so we can easily subtract using the parallel transport. In order to to properly define this derivative, we need to move a vector on the surface, and then project it:

$$\frac{\partial V^a(x)}{\partial x^\mu} = \lim_{dx\to 0} \frac{V^a(x^\mu + (dx)^\mu) - \tilde{V}^a(x^\mu)}{(dx)^\mu}$$

As we attempt to define such an operation, we want the following properties:

$$\tilde{v}^a - v^a \propto \Delta x$$

$$\widetilde{(v+w)} = \tilde{v} + \tilde{w}$$

We can say:

$$\tilde{v}^a - v^a = (\Delta x)^\mu \omega_{\mu b}{}^a \ (v)^b$$

Where the object ω is our connector. We need to allow it to do some infinitesimal rotation. So we can use our rotation generators, giving us the form:

$$\tilde{v}_a - v_a = -\frac{1}{2}(\Delta x)^\mu \omega_\mu{}^{cd} [R(J_{cd})]_a{}^b v_b$$

So we need this ω to allow for certain rotations.

Now we take a brief detour to talk about Young Tableauxs. In d dimensions we start with a box with a d. If we go to the right (symmetric) we go to the right, with each box on the right containing $d+1$. If it's antisymmetric we go down, writing $d-1$. Then we draw the 'hooks' going through the boxes in all possible ways, and our number of possible functions is the product of the boxes, divided by the number of hooks.

So the metric $g_{\mu\nu}$ in d dimensions has:

$$n = \frac{d(d+1)}{2}$$

Many functions. But the frames have d^2 many functions. So there are redundancies in the frame, corresponding to the group $SO(d)$, which we can see in that:

$$d^2 - \frac{1}{2}d(d+1) = \frac{1}{2}d(d-1)$$

Which is the dimension of $SO(d)$.

We can write the metric as:

$$g_{\mu\nu} = \delta_{ad} e_\mu{}^a e_v{}^d$$

And rotations preserve the inner product since:

$$R^T R = \mathbb{I}$$

So:

$$R_a{}^b \delta_{bc} R_d{}^c = \delta_{ad}$$

And so any rotation of a frame operator:

$$\tilde{e} = Re$$

is equally valid.

Now we return to our derivative, which we can rename:

$$\begin{aligned}
\nabla_\mu V^a(x) &= \lim_{\Delta x \to 0} \frac{V^a(x^\mu + (\Delta x)^\mu) - \tilde{V}^a(x)}{(\Delta x)^\mu} \\
\nabla_\mu v_a &= \partial_\mu v_a + \omega_\mu{}^{cd} R(J_{cd})_a{}^b v_b \\
\nabla_v v_a &= e_b^\mu \partial_\mu v_a + \frac{1}{2} e_b{}^\mu \omega_\mu{}^{cd} (J_{cd})_a{}^b v_b
\end{aligned}$$

And so we call this object the **covariant derivative operator**, which acts on objects with down indices:

$$\nabla_a v_b = e_a v_b + \frac{1}{2} \omega_a{}^{cd} (J_{cd})_b{}^f v_f$$

The object ω is called the **spin connection**:

$$\omega_a{}^{cd} = e_a{}^\mu \omega_\mu{}^{cd}$$

Note than in our reference to rotations, we made the choice to use the Euclidean metric. This isn't necessarily always the case, for example in special relativity we would need to consider instead the Minkowski metric. We can then define the derivative on a vector with an up index:

$$\nabla_a v^b = e_a v^b - \frac{1}{2} \omega_a{}^{cd} v^f (J_{cd})_f{}^b$$

Note that our Js are the representations of the generators, representing infinitesimal rotations. Recall that the generators are a Lie algebra, which we can expand:

$$vR^{-1}(g) = [R(\dots)]_c{}^d = [\mathbb{I} - i\theta^{cd} R(J_{cd}) + \dots]^{-1}{}_c{}^d = \mathbb{I}_c{}^d - i\theta^{kl}(-R(K_{kl}))_c{}^d$$

And so the inverse of our generator is just the negative of it, hence the change in signs above.

Homework Problem: Calculate the following:

1. $\nabla_a(T_{bc})$
2. $\nabla_a(T_b{}^c)$

3. $\nabla_a(T^{bc})$
4. $\nabla_a(S_{b_1,\dots b_p}{}^{c_1,\dots c_q})$
5. $\nabla_a(T_b S)c)$
6. $[\nabla_a, \nabla_b]v_c$

Also note that in 2 dimensions, we only have one generator of rotations, which greatly simplifies things. We can consider the example first in 2d:

$$\nabla_a = e_a + \omega_a R(J)$$

And of course we have:

$$[J, J] = 0$$

Now we can recall the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;\; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

And we can then define:

$$\gamma^0 \equiv i\sigma^2$$

$$\gamma^1 \equiv \sigma^1$$

We can consider the anticommutator:

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}\mathbb{I}$$

Where η is the Minkowski metric:

$$\alpha = \beta = 0 \;\Rightarrow\; \eta^{\alpha\beta} = -1$$

$$\alpha = \beta = 1 \;\Rightarrow\; \eta^{\alpha\beta} = 1$$

$$\alpha \neq \beta \;\Rightarrow\; \eta^{\alpha\beta} = 0$$

Alternatively, we can instead define:

$$\gamma^0 = \sigma^3$$

$$\gamma^1 = \sigma^1$$

Which gives anticommutators:

$$\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}\mathbb{I}$$

We can think about these two representations as corresponding to fermions. Majorana fermions are the case of entirely real matrices.

A note on fermions: recall that we started out by constructing generators for rotations about the z axis:

$$\mathscr{R}(i\alpha\frac{1}{2}\sigma^3) = \cos(\frac{1}{2}\alpha)\mathbb{I} + i\sin(\frac{\alpha}{2})\sigma^3$$

When we do a rotation of 360°, we get to the same place but with a minus sign. That is precisely what a fermion is.

We can also consider the case:

$$\mathscr{R}(i\alpha\sigma^1) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Which is another representation of rotations of vectors in 2d, which gives the same vector after a rotation by 360°.

Now we return to our derivative:

$$\nabla_a = e_a + \omega_a R(J)$$

Note that we drop the factor of $\frac{1}{2}$ since there are not indices to worry about double counting. Applying this to a vector gives:

$$\nabla_a v_b = [e_a + \omega_a R(J)]v_b$$

We can construct the following object, which is a doublet on which the Pauli matrices act:

$$\Psi_A = \begin{bmatrix} \Psi_+ \\ \Psi_- \end{bmatrix}$$

And then apply the derivative:

$$\nabla_a \Psi_A = [e_a + \omega_a R(J)]\Psi_A$$

When we act with $R(J)$ on v_b, we get:

$$R(J)v_b = \epsilon_{bc}v_c$$

What if we instead apply it to Ψ_A?

$$R(J)\Psi_A = ?$$

Remember from above that for the Majorana fermions, we had generators:

$$\gamma^0 = \sigma^2 \ , \ \gamma^2 = \sigma^1$$

We then know that:

$$R(J) = \frac{1}{2}\epsilon^{ab}[\gamma_a, \gamma_b]\alpha i\sigma^2$$

Which works out since $i\sigma^2$ gives a real matrix. And so:

$$R(J)\Psi_A = \frac{i}{2}(\sigma^2)_{AB}\Psi_B$$

We can also find that:

$$\nabla_a \phi = [e_a + \omega_a R(J)]\phi = e_a \phi$$

Since the rotation matrices cannot rotate an object with no indices.

Homework problem: Compute the following using the identities we derived above:

$$\nabla_a(v^b v_b) = 0$$

$$[\nabla_a, \nabla_b]\phi = ?$$

$$[\nabla_a, \nabla_b]\Psi_A = ?$$

$$[\nabla_a, \nabla_b]v_c = ?$$

9.5 Maxwell's equations and relativity

Maxwell's equations in 'God units':

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho \ \ , \ \ \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J} \ \ , \ \ \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

Where:

$$\vec{D} = \vec{f}_1(\vec{E}, \vec{B}) \ \ , \ \ \vec{H} = \vec{f}_2(\vec{E}, \vec{B})$$

These functions $\vec{D}$ and $\vec{H}$ get complicated and depend a lot on the material in question. We also have the conservation law:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

These equations also imply:

$$\vec{B} = \vec{\nabla} \times \vec{A} \ \ , \ \ \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

And now some mechanics. Recall that we started by discussing particles as position vectors:

$$\vec{r}(t)$$

And then built up to velocity:

$$\frac{d\vec{r}}{dt} = \vec{v}$$

And then momentum and force:

$$\vec{p} = m\vec{v} \;\; , \;\; \frac{d}{dt}\vec{p} = \vec{F}$$

If we offset our origin, these derivatives stay the same. So we have a symmetry in these basic equations that we often exploit. This symmetry is in the definition of velocity:

$$\vec{r} \rightarrow \vec{r} + \vec{r}_0$$

Also note that in Newton's second law, two derivatives determine the force. Similarly, if we look at Maxwell's equations, we also see two derivatives. So is there a symmetry there as well? Does a symmetry appear when we take the first derivative of the potentials? Yes. We can immediately see that $\vec{B}$ has a symmetry:

$$\vec{A}_2 \rightarrow \vec{A}_1 + \vec{\nabla}\Lambda$$

We want to find a similar transformation of the equation for $\vec{E}$ that similarly leaves it invariant. We have:

$$\vec{E} = -\vec{\lambda}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$$
$$\vec{E} = -\vec{\nabla} - \frac{1}{c}\frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\Lambda)$$
$$\vec{E} = -\vec{\nabla}(\Phi + \frac{1}{c}\frac{\partial}{\partial t}\Lambda) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$$

This is called a gauge symmetry. There is another interesting hidden symmetry in Maxwell's equations: they are invariant under Lorentz transformations.

We know that the universe is described by time and space, so we can write:

$$x^\mu = (ct, \vec{x})$$

We can notice in Maxwell's equations that we have the following:

$$\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$$

$$A_\mu = (\Phi, \vec{A})$$

And so this hints at there being some sort of four dimensional geometry in Maxwell's equations. We can use a structure nearly identical to the L generators to create a matrix representation of Maxwell's equations in four dimensional Minkowski space:

$$F_{ab} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_z & 0 & -B_z & B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & -B_y & -B_x & 0 \end{bmatrix}$$

$$\eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x^a = \begin{bmatrix} ct \\ \vec{x} \end{bmatrix} \quad , \quad \partial_a = \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \vec{\nabla} \end{bmatrix} \quad , \quad A_a = \begin{bmatrix} \Phi \\ -\vec{A} \end{bmatrix} \quad , \quad J_a = \begin{bmatrix} c\rho \\ -\vec{J} \end{bmatrix}$$

Now we are going to define a covariant derivative for $E\&M$:

$$\nabla_\mu = \partial_\mu - iA_\mu \mathscr{R}(Q)$$

Homework problem: Calculate the following:

$$[\nabla_\mu, \nabla_\nu]\Phi = ?$$

Where Φ is an arbitrary complex valued function. Use the fact that:

$$\mathscr{R}(Q)\Phi = e_0$$

10 June 12

10.1 Homework

Yesterday we met the covariant derivative of electromagnetism. What is the commutator of them?

$$[\nabla_\mu, \nabla_\nu]\Phi = -ie_0(\partial_\mu A_\nu - \partial_\nu A_\mu)\Phi$$

And for the general covariant derivative on an arbitrary vector, we got a mess. Dennis and Mina figured it out (thank god Konstantinos had mercy on me).

$$[\nabla_a, \nabla_b]v_c = \left(c_{ab}{}^k + \frac{1}{2}\omega_{[a}{}^{ij}(J_{ij})_{b]}{}^k \right) \nabla_k v_c + \left(C_{ab}{}^k \frac{1}{2}\omega_k{}^{rq} + e_{[a}\frac{1}{2}\omega_{b]}{}^{pq} + \frac{1}{4}(\omega_b{}^{ij}\omega_a{}^{xy}C_{xyij}{}^{pq}) \right) (J_{pq})_c{}^f v_f$$

A note on Mario's (any my) mistake: remember that J is an operator, and can act on the frame operators. We cannot assume that J and e commute. We call these Lie algebra valued operators: they act on anything with appropriate indices. This is why a scalar (even a scalar function) gives zero; they have no indices.

We can now consider:

$$\nabla_a \eta_{bc} = e_a \eta_{bc} + \frac{1}{2}\omega_a{}^{kl}((J_{kl})_b{}^f \eta_{fc} + (J_{kl})_c{}^f \eta_{bf}) = 0$$

The first term must be zero, since e is a differential operator, and η is a constant. What is J? We should be able to explicitly write it if we have a particular Lie algebra. Recall the example from before, where we had a table of numerical values and used them to construct a matrix representation.

With a lot of pain, we found something interesting about the covariant derivative acting on the Minkowski metric: it's zero, because the Minkowski metric is invariant under Lorentz transformations. We can also consider the Levi-Civita tensor, and do a similar annoying process, and get a sum of more terms, since we have more indices. A Lie algebra valued generator always obeys the Leibniz rule (product rule for derivative). If you have a symmetry group and a set of invariants (e.g. Kronecker delta, Levi-Civita), if you construct a covariant derivative with a Lie algebra valued generator in it, it will annihilate each of those invariants.

"What was that video game with the Mario in it?" — Jim

Say we have the Lie algebra of the Lorentz group:

$$[M_{ab}, P_c] = c_{abc}{}^f P_f = i(\eta_{bc} P_a - \eta_{ac} P_b)$$

Where the P_c term has a second fixed index. And so:

$$c_{abc}{}^f = i\left(\delta_a{}^f \eta_{bc} - \delta_b{}^f \eta_{ac}\right)$$

And so we can find a matrix representation of our generator:

$$(J_{ab})_c{}^f = -c_{abc}{}^f$$

Hence:

$$(J_{k\ell})_b{}^f = -i(\delta_k{}^f \eta_{\ell c} - \delta_\ell{}^f \eta_{kc}) = -i\delta_{[k}{}^f \eta_{\ell]b}$$

Recall the equation:

$$[\nabla_a, \nabla_b] v_c = \left(c_{ab}{}^k + \frac{1}{2}\omega_{[a}{}^{ij}(J_{ij})_{b]}{}^k\right)\nabla_k + \left(-C_{ab}{}^k \frac{1}{2}\omega_k{}^{rq} + e_{[a}\frac{1}{2}\omega_{b]}{}^{pq} + \frac{1}{4}(\omega_b{}^{ij}\omega_a{}^{xy} C_{xyij}{}^{pq})\right)(J_{pq})_c{}^f$$

Here the first term is called **torsion** and is of the form $T_{ab}{}^k$ and the second term is the **Riemann curvature tensor**, and has the form $R_{ab}{}^{pq}$. These ω terms must be defined in some way. We can note that the torsion is linear with respect to ω, but the Riemann curvature tensor is quadratic.

Homework problem:

1. We have explicit expression for J. Explicitly calculate the product in the expression:
$$\omega_a{}^{ij}(J_{ij})_b{}^k = ?$$

2. Set torsion to zero and solve for ω. This is a particular choice of connection that we call **spin connection**. We want an explicit description of $\omega_a{}^{ij}$

3. Using the spin connection found above, the commutator becomes just the Riemann curvature tensor. Calculate:

$$[\nabla_a, [\nabla_b, \nabla_c]] = ?$$

Which we have seen before in the Jacobi identity.

Now to discuss the question with the fermions:

$$[\nabla_a, \nabla_b]\Psi_A = [a_a, e_b]\Psi_A + \frac{i}{2}(e_a\omega_b(\sigma^2)_{AB}\Psi_b - e_b\omega_a(\sigma^2)_{AB}\Psi_B)$$

10.2 Some more $E\&SM$

We discussed how Maxwell's equations are closely related to Newton's second law. One of our big questions is if there is something in gravity that similarly reminds us of the second law.

We are going to look at some equations (in Minkowski space) and convince ourselves they are really Maxwell's equations. Recall that we denoted:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

We will also say that (pay attention to the change in notation):

$$A_\mu = \begin{bmatrix} V \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} V \\ \vec{A} \end{bmatrix}$$

And so of course:

$$A_0 = V \ \ , \ \ A_1 = A_1 = A_x \ \ , \ \ A_2 = A_2 = A_y \ \ , \ \ A_3 = A_3 = A_z$$

So we also expect ∂_μ must be a column object as well:

$$\partial_\mu = \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ a_0\frac{\partial}{\partial x} \\ a_0\frac{\partial}{\partial y} \\ a_0\frac{\partial}{\partial z} \end{bmatrix}$$

Note that F is antisymmetric:

$$F_{\mu\nu} = -F_{\nu\mu}$$

So we can use the Young Tableaux to figure out the number of entries we need to figure out:

$$n = \frac{4 \cdot 3}{1 \cdot 2} = 6$$

We also know that the diagonal of the matrix must be zeros, such that the antisymmetry is satisfied. We can then use our formula to find the elements of the matrix:

$$F_{\mu\nu} = \begin{bmatrix} 0 & \frac{1}{c}\frac{\partial A_1}{\partial t} - a_0\frac{\partial V}{\partial x} & \frac{1}{c}\frac{\partial A_2}{\partial t} - a_0\frac{\partial V}{\partial y} & \frac{1}{c}\frac{\partial A_3}{\partial t} - a_0\frac{\partial V}{\partial z} \\ -\frac{1}{c}\frac{\partial A_1}{\partial t} + a_0\frac{\partial V}{\partial x} & 0 & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} & \frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \\ -\frac{1}{c}\frac{\partial A_2}{\partial t} + a_0\frac{\partial V}{\partial y} & -\frac{\partial A_2}{\partial x} + \frac{\partial A_1}{\partial y} & 0 & \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ -\frac{1}{c}\frac{\partial A_3}{\partial t} + a_0\frac{\partial V}{\partial z} & -\frac{\partial A_3}{\partial x} + a_0\frac{\partial A_1}{\partial z} & -\frac{\partial A_3}{\partial y} + a_0\frac{\partial A_2}{\partial z} & 0 \end{bmatrix}$$

We note that the first column can be written as:

$$C_1 = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} + a_0\vec{\nabla}V$$

And the first row as:

$$R_1 = \frac{1}{c}\frac{\partial}{\partial t}\vec{A} - a_0\vec{\nabla}V$$

Similarly, the other entries look suspiciously similar to the curl: “The usual rule is you use this uh witchcraft recipe”

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

And so we have the following entries as:

$$F_{12} = (\vec{\nabla} \times \vec{A})_z$$

$$F_{13} = -(\vec{\nabla} \times \vec{A})_y$$

$$F_{23} = (\vec{\nabla} \times \vec{A})_x$$

And so we know that $a_0 = 1$, since that gives a nice result with Maxwell's equations. And so gives the matrix:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

This object is sometimes called the **Maxwell tensor**. What this reveals is that electromagnetism can be thought of as an analogue of mechanics. We can think of E and B as velocities, the potentials as position vectors, and Maxwell's equations as Newton's second law. The other thing that is special here, is that Maxwell's equations are consistent with special relativity. Or, as Jim says, they are the parents of special relativity.

Now we suppose we consider the following expressions with maximal saturation of indices:

$$0 = \epsilon^{\mu\nu\sigma\rho}\partial_\sigma F_{\mu\nu}$$

$$\eta^{\mu\nu}\partial_\mu F_{\nu\sigma} = 0$$

We showed explicitly in class, that in this geometry, we can state all four of Maxwell's equations as just:

$$\epsilon^{\mu\nu\sigma\rho}\partial_\sigma F_{\mu\nu} = 0$$

$$\eta^{\mu\nu}\partial_\mu F_{\nu\sigma} = \frac{4\pi}{c}J_\sigma$$

From here we can also get the equation:

$$\nabla \cdot \vec{J} = \frac{\partial}{\partial t}\rho$$

The $F_{\mu\nu}$ tensor is a $0, 2$ tensor: it takes two vectors and maps them to a number. A special sub-case of this is called an **n-form**:

$$\omega(v_{p(1)}, v_{p(2)}, \ldots, v_{p(n)}) = \mathrm{sgn}(P)\omega(v_1, v_2, \ldots, v_n)$$

I.e. an $n, 0$ tensor which gives the same result (up to sign) under permutation of input vectors, with sign determined by the sign of the permutation. The F tensor is an $n-form$ since it is antisymmetric under exchange of its two indices. We say that a top form is an n form where $n = d$, the dimension of the space. We can also define a volume form, which is just a top form?

Homework problem: Consider d dimensions, and ϕ an endomorphism. Calculate:

$$\frac{\mathrm{Vol}(\phi(e_1), \phi(e_2), \ldots, \phi(e_d))}{\mathrm{Vol}(e_1, e_2, \ldots, e_d)} = ?$$

11 June 15

11.1 Homework

We went over the homework problems, Konstantinos was very upset with AI use. Problem with the Jacobi identity commutator was complicated, I am not sure what we were even trying to prove.

11.2 After lunch

Some notes on (anti) symmetrization:

$$A_{[m}B_{n]} = A_m B_n - A_n B_m$$

$$A_{(m}B_{p)} = A_m B_p + A_p B_m$$

Similarly the (anti) symmetrization of 3 indices would give 6 terms. The anti symmetrization has alternating signs, the symmetrization is all positive. We can use this to show that the Jacobi identity must hold:

$$\begin{aligned}
\partial_{[m}\partial_n C_{p]} &= \partial_m\partial_n A_p - \partial_n\partial_m A_p + \partial_n\partial_p A_m - \partial_p\partial_n A_m + \partial_p\partial_m A_n - \partial_m\partial_p A_n \\
&= \partial_m(\partial_n A_p - \partial_p A_n) + \partial_n(\partial_p A_m - \partial_m A_p) + \partial_p(\partial_m A_n - \partial_n A_m) \\
&= \partial_m F_{np} + \partial_n F_{pm} + \partial_p F_{mn} \\
&= (\partial_m\partial_n - \partial_n\partial_m)A_p + (\partial_n\partial_p - \partial_p\partial_n)A_m + (\partial_p\partial_m - \partial_m\partial_p)A_n = 0
\end{aligned}$$

Since the partial derivatives commute. So the Jacobi identity must hold. This is what we wanted to show on the homework, using mainly the anti symmetrization we are already comfortable with.

"You should never start a calculation unless you already know the answer" — Warren Siegel, Jim's friend from the '70s

Time for $\mathfrak{su}(3)$. Recall that we have 8 generators for $\mathfrak{su}(3)$ We have a covariant derivative:

$$(\nabla_{\underline{m}})_{\hat{a}}{}^{\hat{b}} = \mathbb{I}\partial_{\underline{m}} + \frac{1}{2} i g_0 A_{\underline{m}}{}^{I} (\lambda_I)_{\hat{a}}{}^{\hat{b}}$$

Where I varies from 1 to 8 (corresponding to the eight generators), and $\hat{a}$ and $\hat{b}$ are $1, 2, 3$, since the Gell-Mann matrices are 3×3. Additionally, m takes on values $0, 1, 2, 3$. We could just as easily have considered the larger adjoint representation of $\mathfrak{su}(3)$. We might suppress some of these indices for simplicity:

$$\nabla_{\underline{m}} = \mathbb{I}\partial_{\underline{m}} + \frac{1}{2} i g_0 A_{\underline{m}}{}^{I} (\lambda_I)$$

Now we want to find the thing that is the analogue of F in electromagnetism, which we can do by taking the commutator:

$$[\nabla_{\underline{m}}, \nabla_{\underline{n}}] = [\mathbb{I}\partial_m + \frac{1}{2} i g_0 A_{\underline{m}}{}^{I} \lambda_I, \mathbb{I}\partial_{\underline{n}} + \frac{1}{2} i g_0 A_{\underline{n}}{}^{J} \lambda_J]$$

Taking great care, we can explicitly compute this to be:

$$[\nabla_m, \nabla_n] = \frac{i g_0}{2} F_{mn}{}^{k} \lambda_k$$

Where:

$$F_{mn}{}^{k} = \left(\partial_{[m} A_{n]}{}^{k} - g_0 A_n{}^{I} A_m{}^{J} f_{IJ}{}^{k}\right)$$

This was a pain, and we had to be very careful of which operators commute. Note that our input vector may depend on our coordinates, as do the As. However the As are specifically scalar valued functions, whereas the λs are 3×3 matrices. The $f2$ are structure constants of the Lie algebra. We compare this to the $E\&M$ case:

$$[\nabla_m, \nabla_n] = -iQF_{mn}$$

Where:

$$F_{mn} = \partial_m A_n - \partial_n A_m$$

We can also do $E\&M$ on a $2d$ curved manifold:

$$\nabla_a = e_a + i\omega_a J + iA_a Q$$

Where Q is a generator corresponding to charge, and J is a rotation generator. The charge Q is a generator of $U(1)$, and J is the single generator of $SU(2)$. In general, every observable can be given a generator. We could also do this in higher dimensions. The electromagnetism itself is always represented by the same $U(1)$ symmetry, but we can change the rotation to describe it on arbitrarily high dimensional manifolds. Similarly, we could consider the standard model on a curved manifold. The standard model is associated with the groups $U(1)$, $SU(2)$, and $SU(3)$. In this case we would sum up terms corresponding to each group:

$$\nabla_a = e_a + \underbrace{\frac{i}{2}\omega_a{}^{ij} J_{ij}}_{\text{Rotation}} + \underbrace{ie_0 A_a Q}_{U(1)} + \underbrace{\frac{ig_y}{2} B_a{}^{\hat{i}} \sigma^{\hat{i}}}_{SU(2)} + \underbrace{\frac{ig_c}{2} C_a{}^{\check{i}} \lambda_{\check{i}}}_{SU(3)}$$

We generally don't care about this however, since the standard model on a curved manifold generally requires quantum gravity, which we have not figured out yet. In the case where there is no curvature (and thus no gravity), all of the ωs are zero, which is just regular quantum mechanics.

Suppose we now consider:

$$T(a) = e^{a\frac{d}{dx}}$$

Which we can use to find:

$$T(a)x = x + a$$

$$T(a)\cos x = \cos(x + a)$$

$$T(a)x^3 = (x + a)^3$$

$$T(a)e^x = e^{x+a}$$

Now let us consider:

$$T(a)f(x) = ?$$

It would seem that:

$$T(a) = f(x + a)$$

And so the function $T(a)$ is translating the function $f(x)$, and so we call it the translation operator, which is often used in quantum mechanics. We can also see that $T(-a)$ is the inverse of $T(a)$, and $T(0)$ is the identity. And hence this forms a (Lie) group.

Now we are returning to talk about rotations. We say that A quantity is an invariant if:

$$\mathcal{L}(x, y, z) = \mathcal{L}(x', y', z')$$

For some coordinate transformation. If we have generators L_1, L_2, L_3, and $\mathcal{L}$ is an invariant, then:

$$i(\alpha L_1 + \beta L_2 + \gamma L_3)\mathcal{L}(x, y, z) = 0$$

We use the notation:

$$\delta_{\mathcal{R}}(\alpha^i) = i\alpha_i L_i$$

Such that:

$$\delta_{\mathcal{R}}\mathcal{L}(x, y, z) = i(\alpha L_1 + \beta L_2 + \gamma L_3)\mathcal{L}(x, y, z)$$

Suppose we have some quantity ε that is measured at $y = 0$ with:

$$\varepsilon(x, y = 0) = A_0 x^4 \equiv \varepsilon_{SM}(x)$$

We can also write ε as a function of x and of y:

$$\varepsilon(x, y) = \varepsilon_{SM}(x) + \varepsilon_{smSM}(x, y)$$

Where:

$$\varepsilon_{smSM}(x, y = 0) = 0$$

Such that the above is still satisfied. We can write it as a Taylor series, but instead of constants we have functions of x:

$$\varepsilon_{smSM}(x, y) = \sum_{n=1}^{\infty} y^n f_n(x)$$

Note that the sum starts at 1 such that the series vanishes at zero. Since ε possesses a symmetry with respect to L_3, it must be the case that $L_3\varepsilon = 0$ such that:

$$\begin{aligned} 0 &= L_3\left(\varepsilon_{SM}(x) + \varepsilon_{smSM}(x, y)\right) \\ &= L_3(\varepsilon_{SM}(x)) + L_3(\varepsilon_{smSM}(x, y)) \\ &= 4iA_0 x^3 y + L_3(\varepsilon_{smSM}(x, y)) \end{aligned}$$

We can apply the L_3 operator to the infinite sum and find that:

$$\varepsilon(x, y) = A_0(x^2 + y^2)^2$$

And hence if we identify a symmetry, we can measure only one part of a system and get information about another.

Homework problem: Suppose $\varepsilon_{SM} = B_0 x^5$ and use the same process to find an expression in terms of x, y, assuming the same symmetry holds.

This is called the **Noether Method** and it can be used in supergravity.

12 June 16

12.1 Notes on homework and what we did yesterday

Consider a similar problem to the one yesterday. The function we want to find:

$$\varepsilon_T(x, y)$$

Is rotationally symmetric about z:

$$\varepsilon_T(x, y) = \varepsilon_T(x', y') = \varepsilon_T(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

For any angle θ. And so we can choose some θ_0 such that:

$$x\sin\theta_0 + y\cos\theta_0 = 0$$

At which point our function becomes:

$$\varepsilon_T(x\cos\theta_0 - y\sin\theta_0, 0)$$

Which must exist since there must be some rotation that takes y to zero. And so we have:

$$f(x) = f(x\cos\theta_0 - y\sin\theta_0) = \varepsilon_T(x, 0)$$

And now let us call:

$$I = x\cos\theta_0 - y\sin\theta_0$$

Consider:

$$I^2 = x^2\cos^2\theta_0 - 2xy\cos\theta_0\sin\theta_0 + y^2\sin^2\theta_0 = x^2\cos^2\theta_0 - xy\cos\theta_0\sin\theta_0 - xy\cos\theta_0\sin\theta_0 + y^2\sin^2\theta_0$$

Which gives:

$$I^2 = x^2\cos^2\theta_0 + y^2\cos^2\theta_0 + x^2\sin^2\theta_0 + y^2\sin^2\theta_0$$

Using the condition of $y' = 0$. Which gives:

$$I^2 = x^2 + y^2$$

And so:

$$f(I) = f(\sqrt{x^2 + y^2})$$

This is called the non perturbative approach, where we calculate directly without using any approximations or Taylor series.

12.2 The more tedious part of introducing you to a few more mathematics

Recall the algebraic structures we discussed: groups, rings, fields, vector spaces. We have not yet discussed the notion of an "algebra".

An **algebra** is an algebraic structure that requires four things:

1. A set A
2. An addition:
$$+ : A \times A \to A$$
3. A scalar multiplication:
$$\cdot : F \times A \to A$$
4. A 'multiplication', which is a map that takes two elements of a vector space and returns another vector:
$$\bullet : A \times A \to A$$

A Lie algebra is an algebra. It is a vector space with a notion of 'multiplication', which is the **called the Lie bracket**. We usually say that the Lie bracket is the commutator. The Lie bracket must be anti-commutative:

$$[A, B] = -[B, A]$$

And satisfy the Jacobi identity. When we move to a representation, each generator A becomes a matrix, and is generally $R(A)$ or $\text{Rep}(A)$. In this case, addition becomes addition of matrices, scalar multiplication becomes regular multiplication of matrices by scalars, and the multiplication (Lie bracket) is the commutator. We can also have 'realizations' of these generators as differential operators instead of matrices. For example:

$$J_{ab} \sim \pm i \left(x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b} \right)$$

Recall our calculation from yesterday:

$$T(a_0) \equiv e^{a_0 \frac{d}{dx}}$$

Which gave:
$$T(a_0) f(x) = f(x + a_0)$$

We called this a **translation**. If we assume $a_0 > 0$, then we have a translation to the left, and if $a_0 < 0$, we have a translation to the right. Note that this can seem somewhat counterintuitive at first. We could also denote our operator as:

$$T(a_0) = e^{-ia_0 \left(-i \frac{\partial}{\partial x}\right)}$$

We call the part in parentheses the generator of translation (in the x direction). Similarly, we can have an operator:

$$\mathcal{H}(a_0) = e^{a_0 \frac{\partial}{\partial t}}$$

And so:
$$\mathcal{H}(a_0) f(t) = f(t + a_0)$$

Which we can think of as 'time evolution', which moves us forward in time (assuming $a_0 > 0$). Which in the physicists convention we would write as:

$$\mathcal{H}(a_0) = e^{-ia_0\left(i\frac{\partial}{\partial t}\right)}$$

Where the part in parentheses is the time translation operator, sometimes called the Hamiltonian. Because momentum is the object that causes translations, we can say that the translation operators are the quantum momentum operators. Similarly, we can think of the time translation operator as corresponding to energy. And so we have two operators:

$$P \rightarrow -i\hbar\frac{\partial}{\partial x}$$

$$E \rightarrow i\hbar\frac{\partial}{\partial t}$$

Note the addition of $\hbar$ factors that will come in handy later. Recall that we classically write:

$$E = \frac{p^2}{2m}$$

And so as an equation of operators:

$$i\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)$$

Where we have added a parameter $\hbar$ for the sake of units. This is the Schrödinger equation. However note that this equation is not consistent with general relativity, since the derivatives of x and t cannot remain in the same form under Lorentz transformations. So our goal now is to improve the Schrödinger equation.

We can first recall Newton's law for a free particle (no force acting on it):

$$\frac{d^2x(t)}{dt^2} = 0$$

But what about for a particle that is described by a function of multiple variables and we want it to be invariant under rotation:

$$\frac{\partial^2\phi(x,y)}{\partial x^2} + \frac{\partial^2\phi(x,y)}{\partial y^2} = 0$$

Similarly, if we want it to be invariant under Lorentz transforms, we get:

$$-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial x^2} = 0$$

Now suppose our particle has a force acting on it, such as a spring:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Then our equation based on Lorentz transforms becomes:

$$-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial x^2}=m^2\phi$$

This equation is called the **Klein-Gordon equation**.

We can play a similar game, and instead rewrite the relation between energy and momentum:

$$E^2=p^2c^2+m^2c^4$$

And then substituting operators gives:

$$-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial x^2}=\left(\frac{mc}{\hbar}\right)^2\phi$$

Which we see has the same form as the wave equation above.

Now what would happen if we use ∂_m to denote our coordinates, with m taking on values 0 for time and 1 for x. We get the equation:

$$\eta^{mn}\partial_n\partial_m\phi=\left(\frac{mc}{\hbar}\right)^2\phi$$

Note that:

$$\eta_{mn}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$$

Is the Minkowski metric, but η^{mn} is the inverse metric, defined such that:

$$\eta^{mn}\eta_{ml}=\delta_l{}^n$$

We can extend this equation to 4 dimensions by allowing for 3 spacial dimensions. The equation stays the same:

$$\eta^{mn}\partial_n\partial_m\phi=\left(\frac{mc}{\hbar}\right)^2\phi$$

Except now m,n can take on values $0,1,2,3$. Also note that that the Minkowski metric and inverse Minkowski metric can sap based on convention, however we will define η_{mn} as the Minkowski metric. We denote:

$$\Box=\eta^{mn}\partial_m\partial_n$$

In five dimensions we use a pentagon, in three dimensions a triangle. So this whole equation becomes:

$$\Box\phi=\left(\frac{mc}{\hbar}\right)^2\phi$$

This method is one way to make quantum mechanics consistent with special relativity. We want to find the other way?

What Dirac did, is he noticed that taking a second order derivative with respect to time leads you to the Klein-Gordon equation. He then asked if there was a way to reduce the spacial derivative instead. He wanted an equation of this form:

$$\gamma^0 \frac{\partial \phi}{\partial t} + \gamma^1 \frac{\partial \phi}{\partial x} = \delta \phi$$

So we can write:

$$\gamma^m \frac{\partial}{\partial x^m} \phi = \delta \phi$$

We can then square the operator on the left to give:

$$\delta^2 = (\gamma^0)^2 \frac{1}{c^2} \partial_{tt} + (\gamma^1)^2 \partial_{xx} + \frac{1}{c} \partial_{xt} (\gamma^0 \gamma^1 + \gamma^1 \gamma^0)$$

Which is the same as the Klein-Gordon equation if the cross term vanishes. So we can find that this equation is satisfied by:

$$\delta = \frac{mc}{\hbar} \mathbb{I}$$

$$\gamma^0 = i\sigma^1$$

$$\gamma^1 = \sigma^2$$

Where the σs are the Pauli matrices, which we know anticommute. Similarly, consider acting on our earlier equation by a differential operator:

$$\frac{\partial}{\partial x^n} \gamma^m \frac{\partial}{\partial x^m} \phi = \frac{\partial}{\partial x^n} \delta \phi$$

$$\gamma^m \frac{\partial}{\partial x^n} \frac{\partial}{\partial x^m} \phi = \delta \frac{\partial}{\partial x^n} \phi$$

Multiplying by γ^n gives:

$$\gamma^n \gamma^m \frac{\partial}{\partial x^n} \frac{\partial}{\partial x^m} \phi = \gamma^n \delta \frac{\partial}{\partial x^n} \phi$$

We make an assumption here that $\gamma^n \delta = \delta \gamma^n$, and using our original equality, this gives:

$$\gamma^n \delta \frac{\partial}{\partial x^n} \phi = \delta^2 \phi$$

On the right side. Now we consider the left side. Call it I:

$$I = \gamma^n \gamma^m \frac{\partial}{\partial x^n} \frac{\partial}{\partial x^m} = \gamma^n \gamma^m \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^n}$$

Noting that partial derivatives commute. And we can change the dummy indices:

$$I = \gamma^m \gamma^n \frac{\partial}{\partial x^n} \frac{\partial}{\partial x^m}$$

And so we can write:

$$2I = (\gamma^n\gamma^m + \gamma^m\gamma^n)\frac{\partial}{\partial x^n}\frac{\partial}{\partial x^m}$$

And hence:

$$I = \frac{1}{2}\{\gamma^m, \gamma^n\}\frac{\partial}{\partial x^n}\frac{\partial}{\partial x^m}$$

Now we will do the same thing in a different way: consider:

$$A^m B^n \frac{\partial}{\partial x^m}\frac{\partial}{\partial x^n}$$

We can write $A^m B^n$ as a symmetric and an antisymmetric component:

$$A^m B^n = \frac{1}{2}A^{[m}B^{n]} + \frac{1}{2}A^{(m}B^{n)} = \frac{1}{2}(A^m B^n - A^n B^m) + \frac{1}{2}(A^m B^n + A^n B^m)$$

So we can write:

$$A^m B^n \frac{\partial}{\partial x^m}\frac{\partial}{\partial x^n} = \frac{1}{2}A^{[m}B^{n]}\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^n} + \frac{1}{2}A^{(m}B^{n)}\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^n}$$

And the only thing that is nonzero is:

$$A^m B^n = \frac{1}{2}A^{(m}B^{n)}\partial_m\partial_n$$

Which gives the same as above in the specific case we are examining.

So now we have the equation:

$$\frac{1}{2}\{\gamma^n, \gamma^m\}\frac{\partial}{\partial x^n}\frac{\partial}{\partial x^m}\phi = \delta^2\phi$$

Comparing this to the Klein-Gordon equation gives:

$$\delta = \pm\frac{mc}{\hbar}$$

And:

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}$$

And due to our assumption from earlier:

$$[\gamma^m, \delta] = 0$$

However these assumptions cannot be satisfied by numbers (or elements of any field). So we need to look elsewhere, at matrices. We can immediately assign $\delta = \mathbb{I}$ which satisfies the necessary condition since it commutes with every matrix. We can see immediately that the Pauli's satisfy this condition (with an extra factor of i). There are of course many solutions, but the smallest dimension matrices we can choose are 2×2, and the Paulis are a convenient option.

However Konstantinos says that there is in fact an even simpler, more economical choice we can make. Recall our equation:

$$\gamma^m \frac{\partial}{\partial x^m}\Psi = \delta\Psi$$

With the conditions:

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}\mathbb{I}$$
$$[\gamma^m, \delta] = 0$$
$$\delta = \pm\frac{mc}{\hbar}\mathbb{I}$$

This forces that Ψ must be a vector. So the equation becomes:

$$(\gamma^m)_\alpha{}^\beta \frac{\partial}{\partial x^m}\Psi_\beta = \pm\frac{mc}{\hbar}\Psi_\alpha$$

Note that we can remove the δ since it is just the identity. We can also write:

$$\Psi = \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}$$

Konstantinos' choice of matrices to use is:

$$\gamma^0 = i\sigma^2$$

$$\gamma^1 = \sigma^1$$

And so both matrices are real, which allows the components of Ψ to be entirely real. Otherwise, we need to consider both real and imaginary components.

The set of objects satisfying these conditions is called a **Clifford algebra**. In a Clifford algebra, the product is the anticommutator.

Sizes of γ matrices in Clifford algebras of D dimensional spacetime:

D (dimension of spacetime)	1	2	3	4	5	6	7	8	9	10	11
d (size of γ matrices)	1	2	2	4	4	8	8	16	16	32	32

We call these matrices **Dirac gamma matrices** when they are complex. When we choose these matrices to be real, we call them the Majorana representation. They always act on spinors.

In all even dimensions, we can define the so called Weyl spinors, which have the form:

$$\Psi = \begin{bmatrix}\psi_1 \\ 0\end{bmatrix}$$

We can have Weyl spinors that are real, which we call Majorana-Weyl spinors. Majorana spinors can be in any number of dimensions. For better or for worse, we live in a 4 dimensional universe, and so we need gamma matrices for 4 dimensions. We do that by taking tensor products:

$$\begin{aligned}
(\gamma^0)_\alpha{}^\beta &= (\sigma^3 \otimes i\sigma^2)_\alpha{}^\beta \\
(\gamma^1)_\alpha{}^\beta &= (\mathbb{I} \otimes \sigma^1)_\alpha{}^\beta \\
(\gamma^2)_\alpha{}^\beta &= -(i\sigma^2 \otimes i\sigma^2)_\alpha{}^\beta \\
(\gamma^3)_\alpha{}^\beta &= (\mathbb{I} \otimes \sigma^3)_\alpha{}^\beta
\end{aligned}$$

Homework problem: Find the explicit matrix representations of these matrices, and verify that they satisfy the conditions for a Clifford algebra.

12.3 Gamma-o-logy

A Clifford algebra has an operation satisfying:

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}\mathbb{I}$$

We can write:

$$\gamma^m\gamma^n = \frac{1}{2}\{\gamma^m, \gamma^n\} + \frac{1}{2}[\gamma^m, \gamma^n] = \eta^{mn}\mathbb{I} + \frac{1}{2}[\gamma^m, \gamma^n]$$

We now define:

$$\sigma^{mn} = \frac{1}{2}[\gamma^m, \gamma^n]$$

We can also consider a product of 3 gammas:

$$\gamma^m\gamma^n\gamma^k = \gamma^m(\eta^{nk}\mathbb{I} + \sigma^{nk}) = \eta^{nk}\gamma^m + \gamma^m\sigma^{nk}$$

We can write:

$$\gamma^{(3)_{mnk}} = \gamma^{[m}\gamma^n\gamma^{k]}$$

And we can continue this further and further according to the number of different gammas we have. In d dimensions, we can antisymmetrize up to d gammas. We could also consider Clifford algebras in Euclidean (or any) space, replacing the Minkowski metric with the Euclidean metric. We often denote a Clifford algebra as $\mathrm{Cl}(p, q)$ where we are in a space with p many temporal and q many spacial dimensions (though the ps and qs may be switched based on convention). So in this example we are in $\mathrm{Cl}(1, 3)$. We generally only consider finite dimensional spaces.

Homework problem: In d dimensions we can define:

$$\gamma^{(l)}_{m_1 m_2 \ldots m_p} = \frac{1}{l!}\sum_{P \in S_l} \mathrm{sgn}(P)\gamma_{m_{P(1)}}\gamma_{m_{P(2)}}\ldots\gamma_{m_{P(l)}}$$

as the antisymmetrization of p many gammas. Now prove:

$$\gamma^{(l)}_{m_1 \dots m_l} = \begin{cases} l \text{ even} & : \ \frac{1}{2}[\gamma_{m_1}, \gamma^{(l-1)}_{m_2 \dots m_l}] \\ l \text{ odd} & : \ \frac{1}{2}\{\gamma_{m_1}, \gamma^{(l-1)}_{m_2 \dots m_l}\} \end{cases}$$

And part two, calculate:

$$\mathrm{Tr}(\gamma^{(2k)}_{m_1 \dots m_{2k}})$$

We can use this notation to write:

$$\gamma^{(4)_{mnkl}} = \epsilon^{mnkl} \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

Since the Levi-Civita symbol gives the sign of the permutation. We can call:

$$i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma^5$$

Not to be confused with γ^m with m varying 0 to 3. For example in the 2 dimensional case:

$$\sigma^0\sigma^1 = (i\sigma^2)(\sigma^2) = \sigma^3$$

We can calculate that:

$$(\gamma^5)^2 = \mathbb{I}$$

Making use of the anticommutation relation:

$$\gamma^a\gamma^b = -\gamma^b\gamma^a$$

So the eigenvalues of γ^5 are ± 1, and we say that spinors corresponding to these eigenvalues have chirality ± 1. Weyl spinners are exactly the spinors that have this propery. We can similarly make use of this property to find that:

$$\gamma^m\gamma^5 = -\gamma^5\gamma^m$$

However this is not true in odd dimensions (they instead commute). We can equivalently say:

$$\{\gamma^m, \gamma^5\} = 0$$

And so we can add γ^5 to our set of γs to get a Clifford algebra in the next higher dimension. Note that we can only do this once, since repeating the same process again gives something proportional to $(\gamma^5)^2 = \mathbb{I}$. Similarly we can evaluate:

$$\epsilon^{mnkl}\gamma_m\gamma_n\gamma_k\gamma_l = -4! \cdot \gamma_0\gamma_1\gamma_2\gamma_3$$

Since the anticommutation relations give the opposite sign as the Levi-Civita symbol (since this is the up indexed Levi-Civita symbol, it is negative). We know that:

$$\gamma_m\eta^{mn} = \gamma^n$$

And:

$$\gamma_0 = -\gamma^0$$

And for $i = 1, 2, 3$:

$$\gamma_i = \gamma^i$$

And hence:

$$\epsilon^{mnkl}\gamma_m\gamma_n\gamma_k\gamma_l = 4! \cdot \gamma^0\gamma^1\gamma^2\gamma^3$$

Be very careful of the difference between up and down indices! Dennis made two mistakes that canceled, but we won't always be that lucky. This is particularly relevant because the Minkowski metric gives a negative sign in certain cases, whereas the Euclidean metric did not, so we could be a bit more cavalier with out notation.

We now want to explore the idea of $\gamma^m\gamma^5$ using our new definition:

$$\gamma^5 = \frac{i}{4!}\epsilon^{mnkl}\gamma_m\gamma_n\gamma_k\gamma_l$$

We found that:

$$\gamma_m\gamma^5 = i\frac{1}{3!}\epsilon_m{}^{nkl}\gamma_n\gamma_k\gamma_l$$

Homework problem: Calculate the identity:

$$\gamma^p\gamma^q\gamma^r = ?$$

In terms of γ, γ_m and γ^5 Also calculate the identity:

$$\sigma^{rs}\gamma^5 = ?$$

And:

$$\gamma_m\gamma_n\gamma^5 = ?$$

And 'if you're really brave':

$$\gamma^{(p)}_{n_1 n_2 \ldots n_p}\gamma^5 = ?$$

Where the answer includes ϵ with d indices and a γ with $(d-p)$ indices.

Now we will introduce a new object. remember that all of the gammas are matrices, with matrix indices:

$$\gamma^m = (\gamma^m)_\alpha{}^\beta$$

We introduce a new object:

$$C_{\alpha\beta} = -i(\sigma^3 \otimes \sigma^2)_\alpha{}^\beta$$

Which has 2 spinorial indices, and no spacetime indices. This object plays the role of a metric for spinors.

Homework problem: Calculate C, and find the number ℓ_1 such that:

$$C_{\alpha\beta} = \ell_1 C_{\beta\alpha}$$

Find the inverse of C, that is find $C^{\alpha\beta}$ such that:

$$C^{\alpha\beta} C_{\alpha\beta} = \delta_\alpha{}^\beta$$

And calculate:

$$(\gamma^m)_{\alpha\beta} = (\gamma^m)_\alpha{}^p C_{p\beta}$$

And find ℓ_3 such that:

$$(\sigma^{mn})_{\alpha\beta} = \ell_3 (\sigma^{mn})_{\beta\alpha}$$

And ℓ_4 such that:

$$(\gamma^m \gamma^5)_{\alpha\beta} = \ell_4 (\gamma^m \gamma^5)_{\beta\alpha}$$

And ℓ_5 such that:

$$(\gamma^5)_{\alpha\beta} = \ell_5 (\gamma^5)_{\beta\alpha}$$

In all of these consider 4 dimensions.

13 June 17

13.1 After I arrived late :(

We see that:

$$\gamma^m \gamma_n \gamma_m = -2\gamma_n$$

We can also find:

$$\mathrm{Tr}(\gamma_m \gamma_n) = \mathrm{Tr}(\eta_{mn} \mathbb{I}) = 4\eta_{mn}$$

Also note that:

$$\mathrm{Tr}(\gamma_p \gamma_q \gamma_r) = 0$$

This follows from the property that if A, B anticommute, then:

$$\mathrm{Tr}(AB) = 0$$

We use the formula from class for the product of 3 γs. What is we consider:

$$\mathrm{Tr}(\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5) = ?$$

Intuitively we expect it to be zero, but why? Because $(\gamma^5)^2 = \mathbb{I}$ we can write:

$$\mathrm{Tr}(\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5) = \mathrm{Tr}(\mathbb{I}\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5) = \mathrm{Tr}(\gamma^5\gamma^5\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5) = \mathrm{Tr}(\gamma^5\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma^5)$$

Which we then similarly permute, get a minus sign, and call it a day. Note that we cannot directly consider anticommutation, since we may have duplicates in our original list, which is why we must introduce γ^5. Takeaway: in dimension

4, the product of any product of an odd number of γs is 0.

The final boss is:

$$\mathrm{Tr}(\gamma_m\gamma_n\gamma_p\gamma_q) = 4(\eta_{mn}\eta_{pq} - \eta_{mp}\eta_{nq} + \eta_{mq}\eta_{np})$$

This is because we can cyclically permute γ_m to the end, and then use the Clifford algebra identity, using linearity, and then doing a similar thing again. Clever.

For homework, calculate:

$$\mathrm{Tr}(\gamma^5)$$

$$\mathrm{Tr}(\gamma^5\gamma_m)$$

$$\mathrm{Tr}(\gamma^5\gamma_m\gamma_n)$$

$$\mathrm{Tr}(\gamma^5\gamma_m\gamma_n\gamma_p)$$

$$\mathrm{Tr}(\gamma^5\gamma_m\gamma_n\gamma_p\gamma_q)$$

And then also calculate:

$$\mathrm{Tr}(\gamma^{(p)}_{\mu_1\ldots\mu_p}\gamma^{(r)v_1\ldots v_r})$$

$$\mathrm{Tr}(\gamma^5\gamma^{(p)}_{\mu_1\ldots\mu_p}\gamma^{(r)v_1\ldots v_r})$$

$$\mathrm{Tr}(\gamma^5\gamma^{(p)}_{\mu_1\ldots\mu_p}\gamma^5\gamma^{(r)v_1\ldots v_r})$$

And also for the set:

$$\Gamma_A = \{\mathbb{I}, \gamma_m, \sigma_{mn}, \gamma_m\gamma^5, \gamma_5\}$$

Calculate:

$$\mathrm{Tr}(\Gamma_A\Gamma_B)$$

13.2 Back to some basics

Consider some vector potential A_m, which has a spacetime index. We can use the metric to get an object with an up index:

$$A^m = A_n\eta^{nm}$$

And similarly:

$$\tilde{A}_n = A^m\eta_{mn}$$

Is this down indexed object the same as the A_m we started with? We see:

$$\tilde{A}_n = A_l\eta^{lm}\eta_{mn} = A_l\delta_n{}^l = A_n$$

This calculation is a bit sloppy, but it works okay since the metric is symmetric. Now consider an object with a spinorial index down: ψ_α. Then:

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta$$

And similarly:

$$\tilde{\psi}_\alpha = \psi^\varepsilon C_{\varepsilon\alpha}$$

Now we want to determine if $\tilde{\psi}_\alpha = \psi_\alpha$. We call this convention "northwest southeast" — raising an index happens from the right (using the rightmost index on C), lowering from the left (using the leftmost index on C). It follows that:

$$\tilde{\psi}_\alpha = C^{\varepsilon\beta}\psi_\beta C_{\varepsilon\alpha} = C^{\varepsilon\beta}C_{\varepsilon\alpha}\psi_\beta = \delta_\alpha{}^\beta\psi_\beta$$

We can prove the following identity:

$$A^m A_m = A_m A^m$$

Because:

$$A^m A_m = A_n\eta^{nm}A_m$$

And:

$$A_n A^n = A_m A^m$$

And so we can do something similar:

$$\psi^\alpha\psi_\alpha = C^{\alpha\beta}\psi_\beta\psi_\alpha = \psi_b C^{\alpha\beta}\psi_\alpha = -\psi_\beta C^{\beta\alpha}\psi_\alpha = -\psi_\beta\psi^\beta = -\psi_\alpha\psi^\alpha$$

And so with spinors, the order in which we contract indices matters. Note we use the fact that C is antisymmetric under exchange of indices. Recall the identity:

$$\mathrm{Tr}(\gamma_m\gamma_n) = 4\eta_{mn}$$

Using the spinorial indices we can write this as:

$$\mathrm{Tr}(\gamma_m\gamma_n) = \delta_\beta{}^\alpha(\gamma_m\gamma_n)_\alpha{}^\beta = (\gamma_m)_\alpha{}^\beta(\gamma_n)_\beta{}^\alpha = 4\eta_{mn}$$

We want to manipulate this object a bit. We can further write:

$$\begin{aligned}(\gamma_m)_\alpha{}^\beta(\gamma_n)_\beta{}^\alpha &= C^{\beta k}(\gamma_m)_{\alpha k}(\gamma_n)^{\delta\alpha}C_{\delta\beta}\\ &= (\gamma_m)_{\alpha k}(\gamma_n)^{\delta\alpha}C^{\beta k}C_{\delta\beta}\\ &= -(\gamma_m)_{\alpha k}(\gamma_n)^{\delta\alpha}C^{\beta k}C_{\beta\delta}\\ &= -(\gamma_m)_{\alpha k}(\gamma_n)^{k\alpha}\delta_\delta{}^k\\ &= 4\eta_{mn}\end{aligned}$$

Taking great care with our treatment of spinorial indices.

14 June 18

Under the operation of taking half traces of Paulis, they behave just like inner products of unit vectors. The identity with the Paulis form a basis for the 2×2 matrices. Similarly, the γ matrices:

$$\mathbb{I}, \gamma^m, \sigma^{mn}, \gamma^5\gamma^m, \gamma^5$$

form a basis for 4×4 matrices. We can find the coefficients by taking traces. We can think of the trace operation on the γ matrices an inner product.

We have the following multiplication table of our objects:

Table of traces of products of Clifford algebra objects

$\mathrm{Tr}(\cdot)$	$\mathbb{I}$	γ_m	σ_{pq}	$\gamma_n\gamma^5$	γ^5
$\mathbb{I}$	4	0	0	0	0
γ_m	0	$4\eta_{mn}$	0	0	0
σ_{mn}	0	0	$4\eta_{pn}\eta_{qm} - 4\eta_{pm}\eta_{qn}$	0	
$\gamma_m\gamma^5$	0		0	$-4\eta_{mn}$	
γ^5	0	0			-4

Konstantinos made me calculate the product of σs by first considering the product of 4 γs, and antisymmetrizing two pairs of indices one after the other. We then see that the commutator emerges, and we can write them as σs, and solve from there. Still not sure where my original approach went wrong, but this one is more elegant I suppose.

Homework problem: Using $\sigma_{mn} = \frac{i}{2}[\gamma_m, \gamma_n]$, we can define some:

$$\Sigma_{mn} = \frac{iC}{2}[\gamma_m, \gamma_n]$$

See if there is a value of C that makes the following look familiar:

$$[\Sigma_{mn}, \Sigma_{rs}] = ?$$

And the follow up is to calculate:

$$[\Sigma_{mn}, \gamma_p] = ?$$

And see if the answer is familiar for any particular C. And then compare these two answers with the algebra of rotations and translations. This is called the Poincare algebra. Going back to our notion of the Clifford algebra objects spanning the 4×4 matrices, we can write any matrix as a linear combination of of these objects, using the trace as the inner product:

$$\begin{aligned}
M_\gamma{}^\delta &= \frac{1}{4}\delta_\beta{}^\alpha(\mathbb{I})_\gamma{}^\delta \\
&+ \frac{1}{4}(\gamma^m)_\beta{}^\alpha(M_\alpha)^\beta(\gamma^m)_\gamma{}^\delta \\
&+ \frac{1}{8}(\sigma_{mn})_\beta{}^\alpha M_\alpha{}^\beta(\sigma^{mn})_\gamma{}^\delta \\
&- \frac{1}{4}(\gamma^5\gamma_m)_\beta{}^\alpha M_\alpha{}^\beta(\gamma^5\gamma^m)_\gamma{}^\delta \\
&+ \frac{1}{4}(\gamma^5)_\beta{}^\alpha M_\alpha{}^\beta(\gamma^5)_\gamma{}^\delta
\end{aligned}$$

We can then write:

$$M_\alpha^\beta = u_\alpha v^\beta$$

And then take partial derivatives:

$$\begin{aligned}
\frac{\partial}{\partial u_k}\frac{\partial}{\partial v^\lambda}M_\gamma{}^\delta = \frac{\partial}{\partial u_k}\frac{\partial}{\partial v^\lambda}u_\alpha v^\beta = \delta_\gamma{}^k\delta_\lambda{}^\delta =& \frac{1}{4}\delta_\lambda{}^k\delta_\gamma{}^\delta + \frac{1}{4}(\gamma_m)_\lambda{}^k(\gamma^m)_\gamma{}^\delta + \frac{1}{8}(\sigma_{mn})_\lambda{}^k(\sigma^{mn})_\gamma{}^\delta \\
&- \frac{1}{4}(\gamma^5\gamma_m)_\lambda{}^k(\gamma^5\gamma_m)_\gamma{}^\delta + \frac{1}{4}(\gamma^5)_\lambda{}^k(\gamma^5)_\gamma{}^\delta
\end{aligned}$$

This is one version of something we call the **Fierz Identity**. Appropriately raising and lowering indices gives another form:

$$\delta_{k\gamma}\delta_\lambda{}^\delta = -\frac{1}{4}C^{\gamma\delta}C_{\lambda k} + \frac{1}{4}(\gamma_m)_{\lambda k}(\gamma^m)^{\gamma\delta} - \frac{1}{8}(\sigma_{mn})_{\lambda k}(\sigma^{mn})^{\gamma\delta} + \frac{1}{4}(\gamma^5\gamma_m)_{\lambda k}(\gamma^5\gamma^m)^{\gamma\delta} - \frac{1}{4}(\gamma^5)_{\lambda k}(\gamma^5)^{\gamma\delta}$$

We start with:

$$C^{\rho\gamma}\delta_\gamma{}^k\delta_\lambda{}^\delta C_{k\epsilon} = C^{\rho\gamma}C_{\gamma\epsilon}\delta_\lambda{}^\delta = -C^{\gamma\rho}C_{\gamma\epsilon}\delta_\lambda{}^\delta = -\delta_\epsilon{}^\rho\delta_\lambda{}^\delta$$

Making use of the fact that the spinor metric is antisymmetric. This is one of the most difficult things to remember. The rest of the switch just comes from appropriately applying the metric to the other side of the first form of the identity. Recall that we define:

$$C^{\delta\alpha} = (i\sigma^3 \otimes \sigma^2)^{\delta\alpha}$$

We saw something similar when we discussed the Paulis spanning the 2×2 matrices. We could also do something similar with the λ matrices.

Homework problem: Do this same calculation with the 8 Gell-Mann matrices and the identity.

Now we want to calculate:

$$\delta_{(k}{}^\gamma\delta_{\lambda)}{}^\delta = -\frac{1}{2}(\gamma_m)_{\lambda k}(\gamma^m)^{\gamma\delta} - \frac{1}{4}(\sigma_{mn})_{\lambda k}(\sigma^{mn})^{\gamma\delta}$$

Since those are the only symmetric terms. Similarly, we can see there are only certain antisymmetric terms, and so:

$$\delta_{[k}{}^\gamma\delta_{\lambda]}{}^\delta = -\frac{1}{2}C_{\lambda k}C^{\gamma\delta} + \frac{1}{2}(\gamma^5\gamma_m)_{\lambda k}(\gamma^5\gamma_m)_{\lambda k}(\gamma^5\gamma^m)^{\gamma\delta} - \frac{1}{2}(\gamma^5)_{\lambda k}(\gamma^5)^{\gamma\delta}$$

14.1 Moving beyond Newton's second law

Consider a volcano erupting in front of a lake. We see both the volcano, and its reflection in the lake. In the lake, the light first travels to the water, then bounces off to reach our eyes. The question is where on the lake does the light ray hit the water. Say the volcano is of height a, we are at height b, and the location of the light hitting the lake x, and say we are a distance d from the volcano. And so the total distance traveled is:

$$D = \sqrt{a^2 + x^2} + \sqrt{b^2 + (d-x)^2}$$

And so we can find the time:

$$T = \frac{D}{c} = \frac{\sqrt{a^2 + x^2} + \sqrt{b^2 + (d-x)^2}}{c}$$

Fermat's principle states that the path taken by a ray of light between two points is the path that takes the least amount of time. And so we have a function $T(x)$, which we can differentiate and set to zero:

$$\frac{dT}{dx} = \frac{1}{c}\left[\frac{x}{\sqrt{a^2+x^2}} - \frac{d-x}{\sqrt{b^2+(d-x)^2}}\right] = 0$$

Which we will notice includes some ratios that can be written as trigonometric functions:

$$\frac{dT}{dx} = \frac{1}{c}\left[\sin\theta_i - \sin\theta_r\right]$$

With i, r corresponding to incidence and refraction. And this is why the angle of incidence and refraction must be zero (Snell's law). However in order to find x we need to solve for x in the uglier equation above, which gives a quadratic with two solutions:

$$x_+ = \frac{ad}{a+b} \ \ , \ \ x_- = \frac{ad}{a-b}$$

Only the first of which is physically reasonable, since the second diverges at $a = b$. This approach does not rely on Newton's second law, but rather a different set of rules for light. In order to see if we are at a minimum or maximum, we can take the second derivative, and see that it is always positive.

Suppose we have some particle without forces acting on it:

$$x(t) = x_0 + v_0 t$$

And we say that at a time T_i it is at point X_i, and at a time T_f it is at point X_f. We can then directly solve for x_0 and v_0 using these parameters. Now we can look at the integral of kinetic energy over our time interval:

$$\int_{T_i}^{T_f} \frac{1}{2}M\left(\frac{dx}{dx}\right)^2 dt = \int_{T_i}^{T_f} \frac{1}{2}M\left(\frac{X_f - X_i}{T_f - T_i}\right) dt = \frac{1}{2}M\frac{(X_f - X_i)^2}{T_f - T_i}$$

We call this quantity **"the action"**. The equations of motion for a trajectory are always such that the action is minimized. This is like Fermat's principle for light, in which the trajectory of light always minimizes the time taken.

For example, we calculated for:

$$x_A(t) = x_i + (x_f - x_i)(\frac{t - T^i}{T_f - T_i})^n$$

$$x_B(t) = x_i + (x_f - x_i)sin(\frac{(4n+1)\pi}{2}\frac{t - T^i}{T_f - T_i})$$

$$x_C(t) = x_i + (x_f - x_i)\frac{1}{e-1}(exp(\frac{t - T_i}{T_f - T_i}) - 1)$$

And found that all of these give an action greater than the path we found a particle 'naturally' takes.

The system we just examined was for a free particle. We can do something similar for a particle that is not free. Our general definition for the action is:

$$S(x(t)) = \int_{t_i}^{t_f} \frac{M}{2}\left(\frac{dx}{dt}\right)^2 dt$$

Imagine that instead of using nature's solution, suppose we perturb the solution slightly:

$$x(t) \to x(t) + \delta_x(t)$$

Which is a function that vanishes at the end points:

$$\delta_x(t_0) = 0$$

$$\delta_x(t_f) = 0$$

We can then calculate the action on this perturbed system:

$$S(x(t) + \delta_x(t)) = \int_{t_i}^{t_f} \frac{1}{2}M\left(\frac{d(x+\delta_x)}{dt}\right)^2 dt$$

And then we can consider the difference:

$$\Delta S = S(x(t) + \delta_x(t)) - S(x(t))$$

We call S a **functional**, since it maps a function to a number. But how do we minimize this? We can recall the familiar formula:

$$\frac{df}{dx} = \lim_{\Delta x \to 0}\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

And so we can say that a change in f is given as:

$$\Delta f \approx \left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)\Delta x$$

So considering this with our example of S gives:

$$\int_{t_i}^{t_f}\frac{1}{2}M\left\{\frac{d}{dt}(x(t)+\delta_x(t))^2\right\}-\int_{t_i}^{t_f}\frac{1}{2}M(\frac{d}{dt}x(t)^2dt \approx \int_{t_i}^{t_f}\frac{1}{2}M\left(\frac{dx}{dt}\right)^2\cdot 2\cdot\frac{d}{dt}\delta_x+\mathcal{O}(\delta_x)$$

And so we say:

$$\Delta S = M_0\int_{t_i}^{t_f}\frac{dx}{dt}\frac{d}{dt}\delta_x dt$$

And so rearranging and using integration by parts gives us:

$$\Delta S = -M\int_{t_i}^{t_f}\frac{d}{dt}\frac{dx}{dt}\delta_x dt+\frac{dx}{dt}\delta_x\Big|_{t=t_i}^{t=t_f} = -M\int_{t_i}^{t_f}\frac{d}{dt}\frac{dx}{dt}\delta_x dt$$

Which forces:

$$M\frac{d}{dt}\frac{dx}{dt}=0$$
$$\delta_x = 0$$

And so the action is the thing that leads to Newton's second law when minimized. And so this object is in many ways more fundamental than Newton's second law is, and when we do physics this is what we care more about. Now we can modify our notion of the action:

$$S(x(t)) = \int_{t_0}^{t_f}\left(\frac{M}{2}\left(\frac{dx}{dt}\right)^2-\frac{1}{2}kx^2\right)dt$$

Using a similar process gives the conditions:

$$\delta_x = 0$$

$$M\frac{d^2}{dt^2}x = -kx$$

Homework problem: do this process for:

$$S = \int_{t_i}^{t_f}\left(\frac{M}{2}\left(\frac{dx}{dt}\right)^2+C_0V\right)dt$$

What about for a free particle in 3 dimensions? We get:

$$S = \int_{t_i}^{t_f}\frac{1}{2}M\left(\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2\right)dt = \int_{t_i}^{t_f}\frac{1}{2}M\delta_{ij}\partial_t x_i\partial_t x_j dt$$

We could swap this to describe a relativistic particle by also considering a fourth dimension, and using the Minkowski metric. Also note that in the case of the relativistic particle the t in the equation just becomes some parameter. Now consider a 2 dimensional particle with a force:

$$S = \int_{t_0}^{t_1} \left(\frac{1}{2} M \left(\left(\frac{\partial x}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial t} \right)^2 \right) - V \right) dt$$

For example in gravity this becomes:

$$S = \int_{t_0}^{t_1} \left(\frac{1}{2} M \left(\left(\frac{\partial x}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial t} \right)^2 \right) - mgy \right) dt$$

Which is the starting point for the famous problem.

Bonus problem: Remember how we generalized Schrödinger's equation to the relativistic case:

$$|\phi = \left(\frac{mc}{\hbar} \right)$$

Where:

$$| = \partial^m \partial^n \eta_{mn}$$

Find the action that gives us this equation when we appropriately integrate over both time and space. Hint: S has the form:

$$S = \int cdt \int dx(\mathcal{L})$$

Where $\mathcal{L}$ depends on ϕ and its derivatives, up to first derivatives appearing up to two times. It is also invariant under Lorentz transformations.

15 June 19

For the last homework problem we have:

$$S = \int \left(\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy \right) dt$$

We can make a substitution:

$$y = f(x)$$

And solve for energy as:

$$E = \frac{1}{2} M(\dot{x}^2 + (f'(x)\dot{x})^2) + mgf(x) = \left[\frac{1}{2}(1 + f'(x)^2)\dot{x}^2 + gf(x) \right] M$$

And so we can say:

$$\left(\frac{dx}{dt} \right)^2 = \frac{\frac{2E}{M} - 2gf(x)}{1 + f'(x)^2}$$

And so:

$$T = \int \sqrt{\frac{1 + y'(x)^2}{\frac{2}{M}(E - mgy)}} dx$$

And so we have a function of the form:

$$T = \int f(y, y')dx$$

So we can substitute:

$$\int f(y + \delta_y, y' + \delta_{y'})dx$$

And so:

$$\Delta S = f(y + \delta_y, y' + \delta_{y'})dx - \int f(y, y')dx = \int \left(\frac{df}{dy}\delta_y + \frac{df}{dy'}\delta_{y'} \right) dx$$

By expanding the sums and canceling terms. And so integration by parts gives:

$$\int \left(\frac{df}{dy}\delta_y - \frac{d}{dt}\left(\frac{df}{dy'} \right) \delta_y \right) dx + \frac{df}{dx}\delta_y \Big|_{\{t_i}^{tf}$$

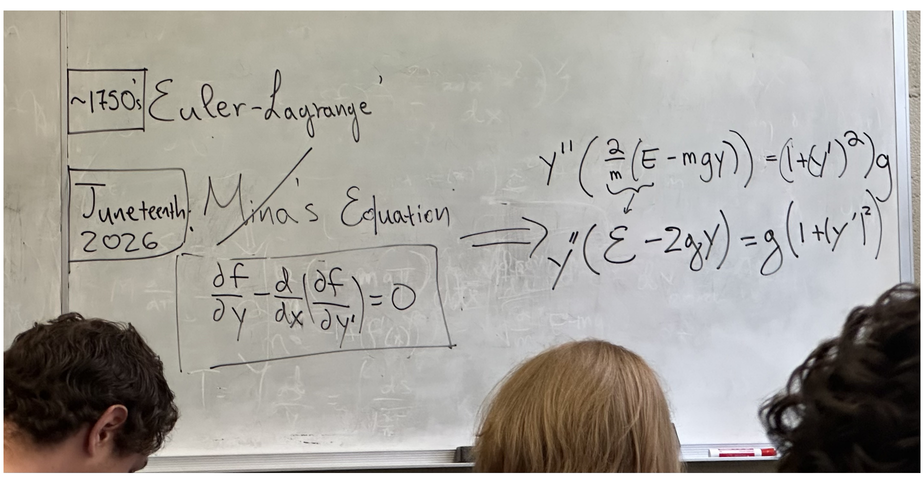
Where the last term is 0. And so we have:

$$0 = \int \left(\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'} \right) \right) \delta_y dx$$

Which forces:

$$\frac{df}{dy} - \frac{d}{dx}\frac{df}{dy'} = 0$$

We call this the **Euler-Lagrange equation**, despite Mina deriving it:



We can then plug our equation for $f(y, y')$ into this equation and get a new differential equation. It involves lot of gross derivatives that should hopefully turn out okay in the end. We get:

$$y''(\varepsilon - 2gy) = g(1 + (y')^2)$$

Where:

$$\varepsilon = \frac{2E}{M}$$

We can examine the units on this equation to see how they work out. We can then solve this differential equation to find the actual equation:

$$\begin{aligned}
y''(\varepsilon - 2gy) &= g(1 + (y')^2) \\
2y'y''(\varepsilon - 2gy) &= 2gy'(1 + (y')^2) \\
\frac{d}{dx}((y')^2)(\varepsilon - 2gy) &= 2gy'(1 + (y')^2) \\
\frac{d}{dx}((y')^2)(\varepsilon - 2gy) &= -\frac{d}{dx}(\varepsilon - 2gy)(1 + (y')^2) \\
\frac{d}{dx}(1 + (y')^2)(\varepsilon - 2gy) &= -\frac{d}{dx}(\varepsilon - 2gy)(1 + (y')^2) \\
\frac{d}{dx}\big((1 + (y')^2)(\varepsilon - 2gy)\big) &= 0 \\
(1 + (y')^2)(\varepsilon - 2gy) &= K_{onstantinos} \\
dx &= \left(\sqrt{\frac{\varepsilon - 2gy}{J_{im} + 2gy}}\right) dy
\end{aligned}$$

Now we let: $\varepsilon - 2gy = \Delta_0 \sin^2 \varphi$

$$\begin{aligned}
dy &= -\frac{\Delta_0}{g} \sin\varphi \cos\varphi d\varphi \\
dx &= \frac{K_{onstantinos}}{g} \sqrt{\frac{K_{onstantinos} \sin^2 \varphi}{K_{onstantinos}(1 - \sin^2 \varphi)}} \sin\varphi \cos\varphi d\varphi \\
x &= \frac{K_{onstantinos}}{2g}\varphi - \frac{K_{onstantinos}}{4g} \sin(2\varphi) + x_0 \\
&= x_0 - \frac{K_{onstantinos}}{4g}(\sin(2\varphi) - 2\varphi)
\end{aligned}$$

It turns out that this is the equation for a cycloid. This problem we have solved is called the **brachistochrone problem**.

15.1 Part two of the day

We can consider the qave equation:

$$\frac{\partial^2 P}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = 0$$

Which we can write as:

$$\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)P = 0$$

We can first consider:

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\check{P}(x,t) = 0$$

We can guess a linear solution of the form:

$$\check{P}(x,t) = a_0 xt + b_0 x$$

And then plugging in gives:

$$b_0 = -a_0$$

What about a quadratic solution?

$$\check{P}(x,t) = a_0 x^2 + b_0 x + a_1 t^2 + b_1 t + c_0 + d_0 ctx$$

Which will then force:

$$\check{P}(x,t) = a_0(ct - x)^2$$

And going to higher orders indicates that we can write:

$$f(x,t) = \sum_{n=0}^{\infty} c_n (ct - x)^n$$

And so this means that the general solution to this differential equation is:

$$P_{(R)}(x,t) = F(xt - x)$$

We then need to do the same thing with the other differential operator in our wave equation. This will give general solutions:

$$P_{(L)}(x,t) = G(ct + x)$$

And so the general solution to this total differential equation is:

$$P(x,t) = F(ct - x) + G(ct + x)$$

Recall our discussion about how the equations of $E\&M$ are largely analogous to Newton's second law. But we now found that Newtonian physics can be derived from action principles. This solution to our differential equation is a wave, not necessarily a particle. Does this same type of action principle also apply to waves? We can consider an action:

$$S = \int_{t_i}^{t_f}\int_{z_i}^{z_f} \mathcal{L}dzdt = \int_{t_i}^{t_f}\int_{z_i}^{z_f}\left(-\frac{1}{2v_0^2}\left(\frac{\partial P}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial P}{\partial z}\right)^2\right)dzdt$$

We can use the same process as yesterday (with the ΔS and integration by parts. Note that are perturbation δ_P can be nonzero, so the other part of the integrand must be zero), which gives the minimum action as the wave equation we found above. So again, this means that we can find Newton's second law from action principles.

Now consider the $1D$ Dirac :

$$i\frac{\partial}{\partial t}\Psi = 0$$

Is there an action that leads to this equation? What if we consider something of the form:

$$S = \int i\Psi\frac{\partial}{\partial t}\Psi dt$$

And then following the same method as before gives:

$$\begin{aligned}
\Delta S &= i\int(\Psi + \delta_\Psi)(\frac{d\Psi}{dt} + \frac{d\delta_\Psi}{dt}) - \Psi\frac{d\Psi}{dt}dt \\
&= i\int \Psi\frac{d\Psi}{dt} + \delta_\Psi\frac{d\delta_\Psi}{dt} + \Psi\frac{d\delta_\Psi}{dt} + \delta_\psi\frac{d\Psi}{dt} - \Psi\frac{d\Psi}{dt}dt \\
&= i\int(-\frac{d\Psi}{dt}\delta_\Psi + \delta_\Psi\frac{d\Psi}{dt})dt = 0
\end{aligned}$$

We can also see that an action of the form:

$$\frac{d^2\Psi}{dt^2} \text{ or } \Psi\Psi$$

Doesn't make sense, since our equation must include exactly one derivative. In order to still use an equation of the form above we need:

$$\Psi\frac{d\Psi}{dt} \neq \frac{d}{dt}(\frac{1}{2}\Psi^2)$$

And so Ψ cannot commute with its derivative. If we instead have them anti-commute:

$$\Psi\frac{d\Psi}{dt} = -\frac{d\Psi}{dt}$$

Then we can have an action of the form:

$$S = \int \frac{i}{2}\Psi\frac{d\Psi}{dt}dt$$

And it works just find, as long as the anticommutation relation is true. We can also derive this anticommutation relation from:

$$\psi(t_1)\psi(t_2) = -\psi(t_2)\psi(t_1)$$

Note that in this 1 dimensional case, the Clifford algebra relations demand that:

$$\{\gamma^0, \gamma^0\} = 2\eta^{00} = -1$$

And hence:

$$\gamma^0 = i$$

We call such functions **anticommuting functions**, such that when you take the product with itself at the same time, it vanishes, but not when you take the products at two different times. We sometimes call these **Grassmann numbers**. These are numbers with the property:

$$\Theta \neq 0 \text{ but } \Theta^2 = 0$$

These numbers anticommute in general. Grassmann functions are anticommuting functions. In order to write an action that produces the Dirac equation, we need Grassmann functions.

We discussed manifolds that are parametrized by real numbers. We could just as easily define a similar object in terms of complex numbers. **Superspace** is s space that can be described only in terms of Grassmann numbers.

"This goose hasn't stopped giving us gifts yet. Golden eggs" — Konstantinos

"Maybe we can take a break because Zeus has been throwing lightning bolts, and these folks have been the recipients, trying to catch them" — Jim

Note that the Grassmann numbers form a ring, but are no longer a field. We can think of general numbers as linear combinations:

$$J = a + b\Theta$$

Which we can also write as a Taylor series expansion:

$$f(x, \Theta) = f(x) + \Theta f_1(x) + \dots$$

Where all higher order terms vanish, since the Θs square to zero. We can think about doing physics in "superspace", where we have ordinance spacetime dimensions, and then another dimension (or more than one other dimensions) of Grassmann numbers. We know in general have two types of functions: our ordinary ones, and our anticommuting ones. Note that the action is generally always a real number. So why do we have an i in our action formula? We have historically learned that:

$$(ab)^\star = b^\star a^\star = a^\star b^\star$$

But for non commuting numbers, we need to be much more careful. For example consider the quaternions $\mathbb{H}$. Any quaternion can be written in the form:

$$x = ai + bj + ck$$

And they are not commutative ($ij = -ji$). And so $i^\star j^\star \neq (ij)^\star$. So in general, complex conjugation changes order, since in general numbers don't need to commute. And so in the case of our action, we have:

$$\left(\Psi\frac{d\Psi}{dt}\right)^{\star}=\left(\frac{d\Psi}{dt}\right)^{\star}\Psi^{\star}$$

Homework problems:

1. In the one dimensional case, can the right hand side of the Dirac equation be nonzero?
$$i\frac{d}{dt}\Psi=m\Psi$$

2. Consider the 2 dimensional Dirac equation (with nonzero right hand side). Find the action.

3. Same thing but 3 dimensions, 4 dimensions is a bonus problem.

16 June 22

16.1 Homework

Note: for spacetime indices we usually use letters from the second half of the alphabet (for example m, n), and for "regular" numbers the first half. So the Dirac equation is:

$$(\gamma^m\partial_m-m)\psi=0$$

Recall our action for 1 dimension:

$$S=\int\frac{i}{2}\psi\partial_t\psi dt$$

If we want to get a term with $m\psi$, we would want a ψ^2 in the action. But we cannot have such a thing, because $\psi^2=0$. And so we can instead write:

$$S=\iint\psi^{\alpha}A_{\alpha}{}^{\epsilon}(\gamma^m)_{\epsilon}{}^{\beta}\partial_m\psi_b dtdx$$

Where A is some specific matrix we can find later. Note that this is without the mass first. We can now make our replacement:

$$\psi\mapsto\psi+\delta\psi$$

Giving:

$$\Delta S=\iint\left(\delta\psi^{\alpha}A_{\alpha}{}^{\epsilon}(\gamma)_{\epsilon}{}^{\beta}\partial_m\psi_{\beta}+\psi^{\alpha}A_{\alpha}{}^{\epsilon}(\gamma^m)_{\epsilon}{}^{\beta}\partial_m\delta\psi_{\beta}\right)dtdx$$

We can then apply integration by parts to give:

$$\Delta S=\iint\left(\delta\psi^{\alpha}A_{\alpha}{}^{\epsilon}(\gamma^m)_{\epsilon}{}^{\beta}\partial_m\psi_{\beta}-\partial_m\psi^{\alpha}A_{\alpha}{}^{\epsilon}(\gamma^m)_{\epsilon}{}^{\beta}\delta\psi_{\beta}\right)dtdx$$

And now we want to bring $\delta\psi$ to the front of the second term, we can use anticommutativity and add an identity matrix to get:

$$\Delta S = \iint \left(\delta\psi^\alpha A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\beta \partial_m \psi_\beta + \delta\psi_\beta A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\gamma \delta_\gamma{}^\beta \partial_m \psi^\alpha\right) dtdx$$

We can then replace the identity with the appropriate metric:

$$\Delta S = \iint \left(\delta\psi^\alpha A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\beta \partial_m \psi_\beta + \delta\psi_\beta A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\gamma C^{\delta\beta} C_{\delta\gamma} \partial_m \psi^\alpha\right) dtdx$$

Where C is the spinor metric. We can apply each metric separately to give:

$$\Delta S = \iint \left(\delta\psi^\alpha A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\beta \partial_m \psi_\beta + \delta\psi^\delta A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\gamma C_{\delta\gamma} \partial_m \psi^\alpha\right) dtdx$$

We can then add another metric to the right term to give:

$$\Delta S = \iint \left(\delta\psi^\alpha A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\beta \partial_m \psi_\beta + \delta\psi^\delta A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\gamma C_{\delta\gamma} C^{\alpha\rho} \partial_m \psi_\rho\right) dtdx$$

We can then apply the metric to give:

$$\Delta S = \iint \left(\delta\psi^\alpha A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\beta \partial_m \psi_\beta + \delta\psi^\delta (-A^{\rho\epsilon})(-(\gamma^m)_{\epsilon\delta}) \partial_m \psi_\rho\right) dtdx$$

Note that we found:

$$C_{\alpha\beta} = (-i\sigma^2)_{\alpha\beta}$$

Recall that objects in our Clifford algebra form a basis:

$$\mathbb{I}, \gamma^0, \gamma^1, \sigma^{01}$$

And so we can exchange indices and write:

$$\Delta S = \iint \left(\delta\psi^\alpha A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\beta \partial_m \psi_\beta + \delta\psi^\delta A^{\rho\epsilon} (\gamma^m)_{\delta\epsilon} \partial_m \psi_\rho\right) dtdx$$

So we can rename indices as:

$$\Delta S = \iint \left(\delta\psi^\alpha A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\beta \partial_m \psi_\beta + \delta\psi^\alpha A^{\beta\epsilon} (\gamma^m)_{\alpha\epsilon} \partial_m \psi_\beta\right) dtdx$$

And so:

$$\Delta S = \iint \left(\delta\psi^\alpha (A^{\beta\epsilon} (\gamma^m)_{\alpha\epsilon} + A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\beta) \partial m \psi_\beta\right) dtdx = 0$$

Which forces:

$$(A^{\beta\epsilon} (\gamma^m)_{\alpha\epsilon} + A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\beta) \partial_m \psi_\beta = 0$$

And so we want A to be such that:

$$A^{\beta\epsilon} (\gamma^m)_{\alpha\epsilon} + A_\alpha{}^\epsilon (\gamma^m)_\epsilon{}^\beta = (\gamma^m)_\alpha{}^\beta$$

We can multiply by $C_{\beta\delta}$ to get:

$$(\gamma^m)_{\alpha\delta} = A_\delta{}^\epsilon(\gamma^m)_{\alpha_\epsilon} + A_\alpha{}^\epsilon(\gamma^m)_{\epsilon\delta}$$

And since γs are symmetric:

$$(\gamma^m)_{\alpha\delta} = A_\delta{}^\epsilon(\gamma^m)_{\alpha_\epsilon} + A_\alpha{}^\epsilon(\gamma^m)_{\epsilon\delta}$$

We want to solve for A in terms of our given basis:

$$A_\delta{}^\epsilon = a\mathbb{I}_\delta{}^\epsilon + b(\gamma^0)_\delta{}^\epsilon + c(\gamma^1)_\delta{}^\epsilon + d(\sigma^{01})_\delta{}^\epsilon = a\delta_\delta{}^\epsilon + b_m(\gamma^m)_\delta{}^\epsilon + c(\sigma^{01})_\delta{}^\epsilon$$

From which we can find:

$$A_\delta{}^\epsilon = \frac{1}{2}\delta_\delta{}^\epsilon$$

Which gives action:

$$S = \int \frac{1}{2}\psi^\alpha(\gamma^m)_\alpha{}^\beta\partial_m\psi_\beta dtdx$$

We can check this by substituting our A into the above expression. Note that the argument depends on the fact that γs are symmetric. This is not always true, but we can often check to find that they are either symmetric or antisymmetric. Now we want to examine the reality of this action. We see that taking the complex conjugate of action gives:

$$S^\star = -S$$

Meaning S is totally imaginary. So we can write instead:

$$S = \int \frac{i}{2}\psi^\alpha(\gamma)_\alpha{}^\beta\partial_m\psi_\beta dtdx$$

This means that we are using the Majorana representation, i.e. we are forcing our matrices and spinors to be real. This simplifies things but is not necessary, nor is it commonplace in most textbooks.

Now suppose we consider:

$$i\psi^\alpha A_\alpha{}^\beta\psi_\beta$$

Complex conjugation gives:

$$(i\psi^\alpha A_\alpha{}^\beta\psi_\beta)^\star = -i\psi_\beta(A_\alpha{}^\beta)^\star\psi^\alpha = i\psi^\alpha(A_\alpha{}^\beta)^\star\psi_\beta$$

And so:

$$(A_\alpha{}^\beta)^\star = A_\alpha{}^\beta$$

We then replace:

$$\psi \mapsto \psi + \delta\psi$$

$$\Delta = i(\delta\psi^\alpha A_\alpha{}^\beta + \psi^\alpha A_\alpha{}^\beta\delta\psi_\beta) = i\delta\psi^\alpha(A_\alpha{}^\beta + (A^T)_\alpha{}^\beta)\psi_\beta$$

This gives the condition:

$$(A + A^T) = -m\mathbb{I}$$

And hence:

$$A = -\frac{m}{2}\mathbb{I}$$

And so our updated action in 2d is:

$$S = \int \frac{i}{2}\psi^\alpha \left((\gamma^m)_\alpha{}^\beta \partial_m \psi_\beta - m\psi_\alpha \right) dtdx$$

Now we want to know what the gauge transformation of the wave function of an electron is:

$$\psi'_\beta = e^{\frac{ie}{\hbar}\alpha(t,\vec{x})}\psi_\beta$$

What is the problem here with the Dirac equation we found? This only works if we can have a complex wave function. This is a fundamental difference between a Dirac spinor and a Majorana spinor. The Majorana spinor cannot be involved in electromagnetic interaction, and the Majorana fermion cannot have charge. Similarly we can consider this in higher dimensions, and without the assumption of things being real (the non-Majorana Dirac equation).

16.2 Supersymmetry at last

Recall that in rotations, the length of the vector is the thing that is invariant. In supersymmetry, the invariant thing is the action. The other peculiar thing: the number of bosonic degrees of freedom, and the number of fermionic degrees of freedom must be exactly the same.

Our starting point is the action for a free particle, which we found to be:

$$S^{(B)} = \int_{t_i}^{t_f} \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 dt$$

However this alone cannot be an invariant, since we also need a version for a fermion. Also note that we often just set $m = 1$ and ignore mass in the equation:

$$S^{(B)} = \int_{t_i}^{t_f} \frac{1}{2} \left(\frac{dx}{dt} \right)^2 dt$$

For a fermion, we have the action:

$$S^{(F)} = \int_{t_i}^{f_f} \frac{i}{2} \Psi \frac{d}{dt} \Psi dt$$

Again, either of these on its own does not give an invariant. Similar to how neither x^2 nor y^2 is an invariant under rotation, but $x^2 + y^2$ is. So the invariant is:

$$S^{SUSY} = S^{(B)} + S^{(F)}$$

We now want to find generators that give rise to these invariants. In rotation, we had invariant:

$$S = x^2 + y^2$$

And generators of the form:

$$\mathbf{L}_3 x = -iy$$

$$\mathbf{L}_3 y = ix$$

These operators are linear in the sense that x, y appear with no powers. We find that an appropriate generator is:

$$\mathbf{L}(x) = \gamma\Psi \ \ , \ \ \mathbf{L}(\Psi) = -i\gamma\frac{dx}{dt}$$

See homework for some additional work on this.

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Using the generator found last class, we can show that $\mathbf{L}(S) = C$, as desired, noting that a constant does not change the physics. The true invariant is the equations of motion we find from minimizing ΔS, which is unaffected by the constance (it always cancels). We can recall that rotations can be written as:

$$R = e^{\alpha\mathbf{L}} = 1 + \alpha\mathbf{L} + \dots$$

Where we can drop the higher order terms. When we apply the rotations to an invariant S, we get:

$$R(S) = S$$

That is, the invariant is unchanged. Using the Taylor series we get:

$$R(S) = (1 + \alpha\mathbf{L})S = S + \alpha\mathbf{L}S = S$$

And so we must have:

$$\mathbf{L}S = C$$

In S^{SUSY}, we have actions corresponding to Bosons ($F = ma$), and Fermions (the Dirac equation). We can vary the action:

$$0 = \Delta S^{SUSY} = S^{SUSY}(x + \delta x) - S^{SUSY}(x)$$

And then the equations of motion minimizing this action will be invariant. We can write this as:

$$\begin{aligned}
\mathbf{L}(S^{SUSY}) &= \mathbf{L}\left(\int dt \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{i}{2}\Psi\frac{d\Psi}{dt}\right) \\
&= \int dt \left(\frac{1}{2}\mathbf{L}\left(\frac{dx}{dt}\right)^2 + \frac{i}{2}\mathbf{L}\left(\Psi\frac{d\Psi}{dt}\right)\right) \\
&= \int dt \left(\frac{dx}{dt}\frac{d\mathbf{L}x}{dt} + \frac{i}{2}\mathbf{L}\left(\Psi\frac{d\Psi}{dt}\right)\right)
\end{aligned}$$

Noting in the last step that $\mathbf{L}$ commutes with the time derivative, and making use of the Leibniz rule. We can now consider the difference between odd and even grassmann numbers. We have odd grassmann numbers with the property:

$$\theta_1\theta_2 = -\theta_2\theta_1$$

Now we can consider products of two grassmann numbers:

$$\rho_{12} = \theta_1\theta_2 \;\;,\;\; \rho_{34} = \theta_3\theta_4$$

And then consider:

$$\rho_{12}\rho_{34} = \rho_{34}\rho_{12}$$

And so we have products of grassmann numbers that commute. Mario calls them "even" grassmann numbers, Konstantinos says they just aren't grassmann numbers. Jim calls it a nilpotent number. We can consider the case where the constant γ is odd.

We can continue simplifying our expression again using the Leibniz rule on the second term, and the fact that we assume $\mathbf{L}$ is an odd grassmann number and γ is even:

$$\mathbf{L}(S^{SUSY}) = \int dt \left(\frac{dx}{dt}\frac{d\mathbf{L}x}{dt} + \frac{i}{2}\left(\mathbf{L}\Psi\frac{d\Psi}{dt} - \Psi\mathbf{L}\frac{d\Psi}{dt} \right) \right)$$

Where for grassmann numbers the Leibniz rule becomes:

$$\mathbf{L}(AB) = \mathbf{L}(A)B + (-1)^G A\mathbf{L}(B)$$

Where:

$$G = \begin{cases} 0 & A \text{ is even and } \mathbf{L} \text{ is odd} \\ 1 & A \text{ is odd and } \mathbf{L} \text{ is odd} \\ 0 & A\ \mathbf{L} \text{ is odd} \end{cases}$$

So plugging in gives:

$$\mathbf{L}(S^{SUSY}) = \int dt \left(\frac{dx}{dt}\gamma\frac{d\Psi}{dt} + \frac{i}{2}\left(\mathbf{L}\Psi\frac{d\Psi}{dt} - \Psi\frac{d\mathbf{L}\Psi}{dt} \right) \right)$$

And so now we integrate by parts to get:

$$\mathbf{L}(S^{SUSY}) = \int dt \left(\frac{dx}{dt}\gamma\frac{d\Psi}{dt} + \frac{i}{2}\left(\mathbf{L}\Psi\frac{d\Psi}{dt} + \frac{d\Psi}{dt}\mathbf{L}\Psi \right) - \frac{d}{dt}(\Psi\mathbf{L}\Psi) \right)$$

Where the last term is just a way of rewriting the boundary term. Because $\mathbf{L}$ and Ψ are both odd, we expect $\mathbf{L}\Psi$ to be even. And so we can commute $\mathbf{L}\Psi$ with the time derivative of Ψ to get:

$$\mathbf{L}(S^{SUSY}) = \int dt \left(\frac{dx}{dt}\gamma\frac{d\Psi}{dt} + i\mathbf{L}\Psi\frac{d\Psi}{dt} - \frac{d}{dt}(\Psi\mathbf{L}\Psi) \right)$$

And so we now see that we should have:

$$\mathbf{L}\Psi = i\gamma\frac{dx}{dt}$$

Which will give:

$$\mathbf{L}(S^{SUSY}) = C$$

Where C is a constant that corresponds to the boundary terms. And so our proposed generators are appropriate.

Recall when we found that:

$$e^{a_0\frac{d}{dx}}f(x) = f(x + a_0)$$

But in general, the derivative is invariant under such coordinate transformations. Now suppose we replace:

$$a_0 \to a(x)$$

This change gives a much more interesting coordinate transformation. And so Einstein's theory is really a gauge theory, that examines these general coordinate transformations. Note: the principle of least action can be better called the 'stability principle' since we don't look to see if the second derivative is up or down. Quantum mechanics is created by putting the action in a complex exponential. This is the path integral, where you consider all possible integrals, which Dirac and Feynman found gives the definitions of quantum mechanics. Feynman defined it as:

$$\iint \mathcal{D}[\Psi]e^{\frac{i}{\hbar}S[\Phi]}$$

Where this is a functional integral integrating over all paths. This is where things get very complicated. Don't think about it too much. We can use this to actually find the Schrodinger equation (see proof in book by Feynman and Hibbs).

We can calculate:

$$\mathbf{L}^2 x = i\frac{dx}{dt}$$

$$\mathbf{L}^2\Psi = i\frac{d\Psi}{dt}$$

Which gives us a Hamiltonian, i.e. the energy operator we saw earlier.

17.1 The Supersymmetry Algebra

In order for the generator to form a closed Lie algebra, we need to add an element, i.e. H, the Hamiltonians. We can consider the commutator:

$$[\mathbf{L}, H] = 0$$

We can also find that:

$$2\mathbf{L}^2 = 2H = \{L, L\}$$

These two identities form the supersymmetry algebra. Since one of our functions is even, we use the anticommutator, if both were odd we could use the commutator. In this case $\mathbf{L}$ is odd and H is even.

Because $\mathbf{L}$ is reserved for rotation generators, we usually write the one dimensional supersymmetry algebra as:

$$A = \{D, H\}$$

$$[H, H] = [D, H] = 0$$

$$\{D, D\} = \kappa_0 H$$

So we can apply the Jacobi identity, applying the appropriate commutators and anticommutators:

$$[H, \{D, D\}] + [D, [D, H]] + [D, [H, D]] = 0$$

We can also appropriately check other combinations of H, D, and find that they are all zero. Note that κ_0 cannot be a grassmann number, since it originates in the anticommutator of D, which is odd, and hence must give an even number. Note that the Jacobi identity is satisfied for any κ_0, but we chose it to be 2 initially by convention. Similarly, we could adjust the constant of γ in our original generators, to get slightly different constants. In general, we will set $\gamma = 1$ and thus have $\kappa_0 = 2$.

And so we have:

$$S^{SUSY} = \int \left[\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + \frac{i}{2} \frac{d\Psi}{dt} \right] dt$$

With a generator D such that:

$$Dx = \Psi \ \ , \ \ D\Psi = i\frac{dx}{dt}$$

Now we make the replacements:

$$\Psi \to \eta \ \ , \ \ \frac{dx}{dt} \to F$$

We can then find the operator:

$$D(\eta) = iF$$

And then consider $D(S)$, and solve as before, using integration by parts again, and we can find that we get the same supersymmetry algebra if we set:

$$D(F) = \frac{d\eta}{dt}$$

Note that η is an odd grassmann number, which requires that we use the appropriate formulation of the Leibniz rule. Also note that F is even (i.e. a 'normal number') and hence it commutes with the time derivative of η. This gives the action:

$$S = \int \left[\frac{i}{2} \eta \frac{d}{dt} \eta + \frac{1}{2} F^2 \right] dt$$

Note that the left term is for a Fermion, and the right one a Boson. A **Fermion** is a field that satisfies the following:

$$\psi(t_1)\psi(t_2) = -\psi(t_2)\psi(t_1)$$

A **Boson** is a field that instead satisfies:

$$x(t_1)x(t_2) = x(t_2)x(t_1)$$

Our replacements would indicate that η is still a fermion. If we vary our new equations of motion, then for the F^2 term we get:

$$F = 0$$

We can talk about the bosonic and fermionic degrees of freedom. We say a degree of freedom is anything that can contribute to the energy of the system. These can be massive particles, photons, or fields like in Maxwell's equations. In our original equation we had 1 bosonic degree of of freedom, and 1 fermionic. But in this case we have lost a bosonic degree of freedom now that we have $F = 0$. This is the equation for a free particle in one dimension. But what if the particle moves? Back in the day we would say:

$$F = ma$$

Taking into account the force. Can we do something similar with supersymmetry? We define an object called the **scalar multiplet**:

$$K = \{x(t), \Psi(t)\}$$

And we can also have a **spinor multiplet**:

$$P = \{\eta, F\}$$

Which tells us that there are at least two systems that exhibit supersymmetry. What if we attach the particle to a spring? We get something very interesting. Also remember that the fermionic term cannot have mass, as the wave function squares to zero. So this particular system is extremely boring. We now consider another action:

$$S^{\text{interaction } SUSY} = \int \left(a_0 i \eta \psi W'(x) + b_0 F W(x) \right) dt$$

First we can act with our generator to decide if it is supersymmetric or not.

Homework problem: We define:

$$[A,B\} = \begin{cases} [A,B] & A \text{ or } B \text{ even} \\ \{A,B\} & A \text{ and } B \text{ odd} \end{cases}$$

Find the <u>correct</u> Jacobi identity! "This one is called the schizophrenic commutator" — Konstantinos

From out above example we get:

$$\begin{aligned}
DS &= \int \left(-a_0 F\psi \frac{dW}{dx} + a_0\eta \frac{dx}{dt}\frac{dW}{dx} + a_0 i\eta\psi D\frac{dW}{dx} + b_0\frac{d\eta}{dt}W + b_0 FDW \right) dt \\
&= \int \left(-a_0 F\psi \frac{dW}{dx} + a_0\eta \frac{dW}{dx} + a_0 i\eta\psi D\frac{dW}{dx} - b_0\eta\frac{dW}{dt} + b_0 FDW \right) dt \\
&= \int \left(-a_0 F\psi \frac{dW}{dx} + (a_0 - b_0)\eta \frac{dW}{dt} + b_0\frac{d}{dt}(\eta W) + a_0 i\eta\psi D\frac{dW}{dx} + b_0 F\psi\frac{dW}{dx} \right) dt
\end{aligned}$$

Since we know that:

$$DW = D(x)\frac{dW}{dx} = \psi\frac{dw}{dx}$$

And so we can collect terms:

$$DS = \int \left(-(a_0 - b_0)F\psi\frac{dW}{dx} + (a_0 - b_0)\eta\frac{dW}{dt} + b_0\frac{d}{dt}(\eta W) \right) dt$$

Since can write:

$$a_0 i\eta\psi D\frac{dW}{dx} = a_0 i\eta\psi D(x)\frac{d^2W}{dx^2} = 0$$

Since $\psi^2 = 0$, thus eliminating that term. Also note that the final term is the total derivative that comes from the integration by parts step. And so this means that we can have any constants as long as:

$$a_0 = b_0$$

And so in that case we have found another supersymmetric action! The most interesting thing about this action.

What we do now is to set the fermionic terms to zero, and calculate the laws of motion:

$$S = \int \left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}F^2 + F(W(x)) \right) dt$$

We vary both x and F and minimize like before, using the same tricks. We get equations of motion:

$$FW'(x) = \frac{d^2x}{dt^2} \quad , \quad F = -W(x)$$

And so:

$$-\frac{1}{2}\frac{d}{dx}W(x)^2 = \frac{d^2}{dx^2}$$

And we can call W the **superpotential**, where:

$$V = \frac{1}{2}W(x)^2$$

Giving a familiar equation:

$$-\frac{d}{dx}V = a$$

Now we want to try this with a spring. We find that we get:

$$W(x) = \omega x$$

In supersymmetry, you need bosons and fermions, sort of like how to have rotational symmetry you need x and y. We can then plug in this W to the sum of the fermionic terms, and find the motion equations, which give:

$$\frac{d\psi}{dt} = \omega\eta \ , \ \frac{d\eta}{dt} = -\omega\psi$$

We can solve these differential equations to get:

$$\psi(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\eta(t) = A\cos(\omega t) - B\sin(\omega t)$$

Now we have four variables: x, ψ, η, F, and we know how the D generator acts on all of them. We need to check that the generator applied to these functions satisfies the algebra we laid out earlier. We see that:

$$D^2 \begin{bmatrix} x \\ \psi \\ \eta \\ F \end{bmatrix} = D \begin{bmatrix} \psi \\ i\frac{dx}{dt} \\ iF \\ \frac{d\eta}{dt} \end{bmatrix} = \begin{bmatrix} i\frac{dx}{dt} \\ i\frac{d\psi}{dt} \\ i\frac{d\eta}{dt} \\ i\frac{dF}{dt} \end{bmatrix} = H \begin{bmatrix} x \\ \psi \\ \eta \\ F \end{bmatrix}$$

It similarly follows that: $[H, D] = [H, H] = 0$. However if we do not include F, then this is not true, and $\{D, D\}$ is not necessarily equal to $2H$, unless we force that η satisfy the Dirac equation:

$$i\frac{d}{dt}\eta = 0$$

Which we call **on-shell closure**. The other case where we have F and do not force this condition on η is **off-shell closure**. We often call F an **auxiliary field** (or auxiliary function). We call it this because it contains no physics, but rather serves as an object to close the algebra.

Homework problem: "$W(x) = A_0 + B_0 x + \frac{1}{2} C_0 x^2$". Say that this is the W, and figure out what the physics connected to it is.

Supersymmetry is a property of a theory that allows us to have a transformation from bosons to fermions that preserves the action (up to a total derivative).

"Alternative solution. If you are in an elevator with people who know nothing about physics, pull up this chart—have it on your phone—and ask them which one looks more balanced. If they say the first one, find them mental help" — Jim

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Jim wants us to watch 'The Paper Chase' (1973).

18.1 Homework

We start with the Schizophrenic commutator. Is there a way to set this up so that we don't have to use cases? We can write the schizophrenic commutator as:

$$[A, B\} = AB - (-1)^{|A||B|} BA$$

Where $|A|$ gives the 'parity' of the grassmann numbers:

$$|A| = \begin{cases} 0 & A \text{ even} \\ 1 & A \text{ odd} \end{cases}$$

We can then use this form to try to figure out how we can rewrite the Jacobi identity:

$$[A, [B, C\}\} = A\left(BC - (-1)^{|B||C|} CB\right) - (-1)^{|A|(|B|+|C|)} \left(BC - (-1)^{|B||C|} CB\right) A$$

How can we use this to get zero when we sum the cyclic permutations of these terms? We can permute the terms:

$$\begin{aligned}
[A, [B, C\}\} + [C, [A, B\}\} + [B, [C, A\}\} = \\
& A\left(BC - (-1)^{|B||C|} CB\right) - (-1)^{|A|(|B|+|C|)} \left(BC - (-1)^{|B||C|} CB\right) A \\
& + B\left(CA - (-1)^{|C||A|} AC\right) - (-1)^{|B|(|C|+|A|)} \left(CA - (-1)^{|C||A|} AC\right) B \\
& + C\left(AB - (-1)^{|A||B|} BA\right) - (-1)^{|C|(|A|+|B|)} \left(AB - (-1)^{|A||B|} BA\right) C \\
& \stackrel{?}{=} 0
\end{aligned}$$

To make this zero, we need to add coefficients to each commutator that depend on $|A|, |B|, |C|$:

$$\alpha[A,[B,C\}\} + \beta[C,[A,B\}\} + \gamma[B,[C,A\}\} = 0$$

We can collect terms in the above expansion to get a system of equations:

$$\alpha - (-1)^{|C|(|A|+|B|)}\beta = 0$$
$$\gamma - (-1)^{|A|(|B|+|C|)}\alpha = 0$$
$$\beta - (-1)^{|B|(|C|+|A|)}\alpha = 0$$

Which gives:

$$\beta = (-1)^{|C||B|}$$
$$\alpha = (-1)^{|A||C|}$$
$$\gamma = (-1)^{|A||B|}$$

Giving a new **super Jacobi identity**:

$$(-1)^{|A||C|}[A,[B,C\}\} + (-1)^{|C||B|}[C,[A,B\}\} + (-1)^{|A||B|}[B,[C,A\}\} = 0$$

Now for the second homework problem, we got the equations of motion:

$$\begin{aligned} \frac{d\psi}{dt} - \eta(B_0 + C_0 x) &= 0 \\ \frac{d\eta}{dt} + \psi(B_0 + C_0 x) &= 0 \\ \frac{d^2x}{dt^2} = -\frac{d}{dx}\left(\frac{W^2}{2}\right) &+ iC_0\eta\psi \\ F &= -W \end{aligned}$$

With the given:

$$W = A_0 + B_0 x + \frac{1}{2}C_0 x^2$$

Solving them however was much more difficult. The only way Jim knows to actually find a solution is to either make some approximations and assumptions, or to find a numerical solution.

Einstein taught us that energy of a particle is something like:

$$E = \pm\sqrt{|\vec{p}|^2c^2 + (mc^2)}$$

We can think of each type of fundamental particles as the vibrational modes in the fundamental fields. The wave function is not the particle in quantum field theory, but rather those oscillations themselves are the particle. Recall the wave equation:

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi + \frac{\partial^2}{\partial x^2}\Phi = 0$$

We have certain vibrational modes: the string being stationary, having 1 'hump', 2 'humps', etc. In field theory, those modes themselves are particles, not the string. Of all the theories we have, this is one of the ones in the best agreement with measurement.

"Particle physics is probably the highest concentration of the craziest math you have thought of, and even the math you haven't thought of" — Jim

18.2 Crazy witchcraft mathematics

Chiral supermultiplet in 4 dimensions, $N = 1$. Spinors are 4 vectors, γ matrices are 4×4, we have 1 D operator.

$$
\begin{aligned}
D_a A &= a_1 \psi_a \\
D_a B &= a_2 (\gamma^5)_a{}^b \psi_b \\
D_a \psi_b &= a_3 i (\gamma^\mu)_{ab} \partial_\mu A - a_4 (\gamma^5 \gamma^\mu)_{ab} \partial_\mu B - a_5 i C_{ab} F + a_6 (\gamma^5)_{ab} G \\
D_a F &= a_7 (\gamma^\mu)_a{}^b \partial_\mu \psi_b \\
D_a G &= a_8 i (\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \psi_b
\end{aligned}
$$

The chiral multiplet has the following set of fields:

$$\{A, B, \psi_a, F, G\}$$

Where A, B, F, G are bosonic, and ψ_a are fermionic. Note that since a varies, we have equally many fermionic and bosonic degrees of freedom. We say that A, B are **dynamical** since their energy is described by differential equations. We say F, G are **auxiliary**, since their equations of motion are entirely algebraic (not differential equations). We can use this information to write the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A \partial^\mu A - \frac{1}{2}\partial_\mu B \partial^\mu B + \frac{1}{2} i (\gamma^\mu)^{cd} \psi_c \partial_\mu \psi_d + \frac{1}{2}F^2 + \frac{1}{2}G^2$$

Note that we could also write the A, B terms with down indexed derivatives if we also include the Minkowski metric:

$$-\frac{1}{2}\partial^\mu A \partial_\mu A = -\frac{1}{2}\partial_\mu A \partial_\nu A \eta^{\mu\nu}$$

The sign is negative, since we want the time derivative coefficient to be positive, and so the rest become negative. The term for the spinor is the term we found before that gives rise to the Dirac equation. It is a Majorana (real) spinor, such that the number of bosonic and fermionic degrees of freedom is the same.

"You know we have a president who falls asleep in meetings, I was wondering if Sammy had gone to the dark side" —Jim

The third term in the Lagrangian is thus:

$$\mathcal{L}_3 = \frac{i}{2}\psi_a(\gamma^\mu)^{cd}\partial_\mu\psi_d$$

The sign here is positive, since we cannot 'fix' the sign issue like we did on the other terms, which have two derivatives. So this one is really just a convention, but we usually take it to be positive (though there are times where the supersymmetry algebra can force certain things).

"You come in here with a skull full of mush, and leave thinking like a ~~lawyer~~ physicist." —Guy in the movie / Jim

Note that we could also change our convention and use the Minkowski metric with time positive and space negative, and we would end up with flipped signs in our kinetic energy terms. The last two terms, corresponding to F, G are just:

$$\mathcal{L}_4 = \frac{1}{2}F^2$$

Again positive, because of how we defined our operators. Before, we saw that in order for us to have 1d interactions we had two fermions and two bosons: x, ψ, η, F. For interactions in 2d, we would expect 8 fields, 4 bosons and 4 fermions: $x, \psi, \eta, F, \tilde{x}, \tilde{\psi}, \tilde{\eta}, \tilde{F}$. These correspond to the 8 fields we have here.

What are the engineering dimensions of our fields? Using the appropriate (natural) units, we can say that energy and mass have the same dimension. We can then us familiar equations to get:

$$\begin{aligned}
[E] &= [M] \\
[x] &= [P]^{-1} \\
[E] &= [t]^{-1} \\
[x] &= [t]
\end{aligned}$$

From the assumption that $c = \hbar = 1$. We can say $[M] = 1$, which gives $[S] = 0$, and $[\mathcal{L}_{4D}] = 4$. We also say that a partial derivative has dimension 1. This forces $[A] = [B]1$ (so that $1 + 1 + 1 + 1 = 4$), and similarly $[\psi] = \frac{3}{2}$, and $[F] = [G] = 2$. Note that since the auxiliary fields have no derivatives, they can be easily recognized by their even dimensions, and fermions by their half integer dimensions. We can do this similarly in arbitrary dimensions.

We can now figure out how the generator D_a acts on each of our fields. We start with $D_a A$. We know it must be a fermion (or its derivative), and since we need to have half integer dimension we must have:

$$D_a A = a_1 \psi$$

Similarly, we know that D_aB must also be a fermion, but can make it distinct from the previous term by adding a γ^5. Like before, dimensions forbid us from adding a derivative or anything like that:

$$D_aB = ia_2(\gamma^5)_a{}^b\psi_b$$

Note the addition of a factor of i to ensure the result is real. We now consider D_aF, which we know must be fermion and include ψ. We said that $[D_a] = \frac{1}{2}$, and so:

$$[D_aF] = \frac{5}{2}$$

And so must the other side of the equation. We can do that by adding a partial derivative:

$$D_aF = a_3(\gamma^\mu)_a{}^b\partial_\mu\psi_b$$

We can write something similar but distinct for D_aG by adding a γ^5:

$$D_aG = ia_4(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi_b$$

We lastly need to find the action of D_a on ψ. We know it must be a boson, and have dimension 2. So we need to get a sum of four terms and have dimension 2. We can construct something appropriately:

$$D_a\psi_b = ia_5(\gamma^\mu)_{ab}\partial_\mu A + a_6(\gamma^5\gamma^\mu)_{ab}\partial_\mu B + ia_7C_{ab}F + a_8(\gamma^5)_a{}^bG$$

Homework problem: Calculate $D_a\mathcal{L}$ and verify it is a total derivative.

19 June 25

Konstantinos: There is no experimental evidence of SUSY in fundamental particle physics, but there is in nuclear physics. That's not the most fundamental thing, but it shows that there might be something to all this.

Jim: He wrote an article in physics today in 2006 (see: https://physicstoday.aip.org/features/is-string-theory-phenomenologically-viable) , when he was asked to comment on if he thought they would find supersymmetry. He said no. We can make predictions using the standard model using Feynman diagrams, particularly ones with loops in them. We can use this to estimate masses of particles. Jim estimated that superpartners would have minimum masses of around 30 TeV, much less the initial energy of the LHC. "Of course nobody listened". Even after the upgrade, the energy they reached was only 14 TeV, and they did not find anything. "A lot of people said 'oh we were promised supersymmetry'. But not if you did that calculation you weren't. But nobody did it. Any competent physicist could have done that calculation". Part of the reason is that such an experiment was very expensive, and showing doubt in the ability to find superpartners would weaken the argument for LHC. Jim doesn't think we will find anything until we have a collider of up to 30 TeV or even 60 TeV. "I'm a theorist, we can't tell our factors of two".

We can also make a graph of mass of particle discovered vs. year they were discovered, and find that on average our ability to discover more massive particles is around 1.5 GeV/year. "You can call this Gates' law if you want, cause no one else has done it". This means we can expect a loooong time to pass before we see direct evidence. Jim only sees one theory that can unify physics: string theory. Jim is a strong defender of string theory. String theory requires supersymmetry in order to be mathematically consistent, and is also the best candidate for quantum gravity. String theory allows for particles of all spins to exist, including the spin 2 graviton. Jim also wrote a paper about how collisions between supersymmetric particles could leave a signature in the cosmic microwave background, which could allow us to discover supersymmetry. Jim also notes that extra dimensions are not necessary for mathematical consistency in string theory.

19.1 Back to calculations

We now verify the homework calculation: invariance of the Lagrangian. We act D_a on $\mathcal{L}$:

$$\begin{aligned} D_a\mathcal{L} = &-\frac{1}{2}\partial^\mu D_a A \partial_m u A - \frac{1}{2}\partial^\mu A \partial_\mu D_a A \\ &- \frac{1}{2}\partial^\mu D_a B \partial_m u b - \frac{1}{2}\partial^\mu B \partial_\mu D_a B \\ &+ \frac{i}{2} D_a \psi_c (\gamma^\mu)^{cd} \partial_\mu \psi_d - \frac{i}{2}\psi_c (\gamma^\mu)^{cd} \partial_\mu D_a \psi_d \\ &+ \frac{1}{2} D_a F F + \frac{1}{2} F D_a F \\ &+ \frac{1}{2} D_a G G + \frac{1}{2} G D_a G \end{aligned}$$

The two A and the two B terms can each be combined, since they are bosons everything commutes. On the fermion term, we can use integration by parts to change the partial derivative.

$$\begin{aligned} D_a\mathcal{L} = &-(\partial^\mu A)(\partial_\mu D_a A) \\ &-(\partial^\mu B)(\partial_\mu D_a B) \\ &+ \frac{i}{2} D_a \psi_c (\gamma^\mu)^{cd} \partial_\mu \psi_d + \frac{i}{2}\partial_\mu \psi_c (\gamma^\mu)^{cd} D_a \psi_d + \partial_\mu \left(-\frac{i}{2}\psi_c (\gamma^\mu)^{cd} D_a \psi_d \right) \\ &+ \frac{1}{2} D_a F F + \frac{1}{2} F D_a F \\ &+ \frac{1}{2} D_a G G + \frac{1}{2} G D_a G \end{aligned}$$

We can then relabel indices on the ψ term and rearrange terms to get:

$$\begin{aligned}D_a\mathcal{L} = &-(\partial^\mu A)(\partial_\mu D_a A)\\ &-(\partial^\mu B)(\partial_\mu D_a B)\\ &+\frac{i}{2}D_a\psi_c(\gamma^\mu)^{cd}\partial_\mu\psi_d+\frac{i}{2}D_a\psi_c(\gamma^\mu)^{cd}\partial_\mu\psi_d+\partial_\mu\left(-\frac{i}{2}\psi_c(\gamma^\mu)^{cd}D_a\psi_d\right)\\ &+\frac{1}{2}D_aFF+\frac{1}{2}FD_aF\\ &+\frac{1}{2}D_aGG+\frac{1}{2}GD_aG\end{aligned}$$

Which then allows us to combine terms as:

$$\begin{aligned}D_a\mathcal{L} = &-(\partial^\mu A)(\partial_\mu D_a A)\\ &-(\partial^\mu B)(\partial_\mu D_a B)\\ &+iD_a\psi_c(\gamma^\mu)^{cd}\partial_\mu\psi_d+\partial_\mu\left(-\frac{i}{2}\psi_c(\gamma^\mu)^{cd}D_a\psi_d\right)\\ &+FD_aF\\ &+GD_aG\end{aligned}$$

Noting that there is a total derivative in the ψ term that comes from integrating by parts. We are now ready to substitute the transformation laws, collect terms, and show that it is actually invariant (up to total derivatives). We use Lots of integrating by parts and making some symmetry arguments. The B terms collected are:

$$-ia_2\partial^\mu B(\gamma^5)_a{}^b\partial_\mu\psi_b-ia_4\partial_\nu B(\gamma^5\gamma^\nu)_{ac}(\gamma^\mu)^{cd}\partial_\mu\psi_d$$

In the second term, we can add an identity and split it into two Cs, and use them to raise and lower indices:

$$(\gamma^5\gamma^\nu)_{ac})(\gamma^m)^{cd}=(\gamma^5\gamma^\nu)_a{}^c(\gamma^\mu)_c{}^d=(\gamma^5\gamma^\nu\gamma^\mu)_a{}^d$$

And we can use the identity from the Clifford algebra to get:

$$(\gamma^5\gamma^\nu\gamma^\mu)_a{}^d=\gamma^5(\frac{1}{2}\{\gamma^\nu,\gamma^\mu\}+\frac{1}{2}[\gamma^\nu,\gamma^\mu])=(\gamma^5\eta^{\nu\mu}+\frac{1}{2}\gamma^5[\gamma^\nu,\gamma^\mu])$$

We can then plug this back into the second term above. The first term will give the same thing as the first term, giving a sum:

$$(-ia_2+ia_4)\partial^\mu B(\gamma^5)_a{}^b\partial_m u\psi_b$$

And then for the second term we can use integration by parts to get the partial derivatives on the same field. We can then note that the partial derivatives use the same indices as are on the γ matrices. These matrices are antisymmetrized, and there is thus an antisymmetry on the ν,μ indices. However we now have two partial derivatives on the same field with the same indices, which gives a symmetry. So this term must be zero, except for the addition of a total derivative arising from the integration by parts:

$$\frac{i}{2}a_4\partial_\nu(G)\gamma^5[\gamma^\nu,\gamma^\mu])_a{}^d\partial_\mu\psi_d)$$

This same process applies similarly for the A terms:

$$(-a_1+a_3)\partial^\mu A\partial_\mu\psi_a+\partial_\mu\left(\frac{a_3}{2}([\gamma^\mu,\gamma^\nu])_a{}^d A\partial_\nu\psi_d\right)$$

The F terms collected are:

$$a_5C_{ac}G(\gamma^\mu)^{cd}\partial_\mu\psi_d+a_7F(\gamma^\mu)_a{}^b\partial_\mu\psi_d=a_5G(\gamma^\mu)_a{}^d\partial_\mu\psi_d(a_7-a_5)$$

Noting that we gain a minus sign when we flip indices on the spinor metric. The process is very similar for the F terms:

$$(a_7-a_5)F(\gamma^\mu)_a{}^d\partial_\mu\psi_d$$

The final term we need to examine is ψ. The terms are:

$$G(ia_8(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi_b)+a_6(\gamma_5)_{ac}G(\gamma^\mu)^{cd}\partial_\mu\psi_d=G(ia_8(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi_b)-Ga_6(\gamma^5\gamma^\mu)_a{}^d\partial_\mu\psi_d$$

Which gives the G terms equal to:

$$i(a_8-a_6)G(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi_b$$

And so all together this forces our constants to be such that:

$$\begin{aligned}a_1&=a_3\\a_2&=a_4\\a_5&=a_7\\a_6&=a_8\end{aligned}$$

Now in order to ensure that we have a supersymmetric algebra, we have to check closure. We require that:

$$\{D_a,D_b\}=2i(\gamma^\mu)_{ab}\frac{\partial}{\partial x^\mu}$$

We add the factor of γ^μ to contract the μ index, and to ensure a result that is symmetric under exchange of a,b. We must write two down indices on the right since we have the same on the left. The coefficient of 2 is just a convention. We also must have the other commutator:

$$[D_a,\partial_\mu]=0$$

Noting that:

$$\partial_\mu=\frac{\partial}{\partial x^\mu}$$

And the constants that precede it in the previous commutator can be thought of as structure constants, and not a part of the actual operator. So our set of generators is $\{D_a, \partial_\mu\}$, i.e. our translation generators and our supersymmetry generators. We could also add to this set $J_{\mu\nu}$, which is our generators of rotations in 4d spacetime, and then examine what happens when we compose rotations with translations and with our supersymmetry generators:

$$[J_{\mu\nu}, D_a] = (\Sigma_{\mu\nu})_a{}^b D_b$$

$$[J_{\mu\nu}, P_\rho] = i\eta_{[\mu|\rho|}P_{\nu]}$$

Note that we care about the commutator since they both contain bosonic terms. We use the anticommutator when considering two fermionic objects. Also recall that:

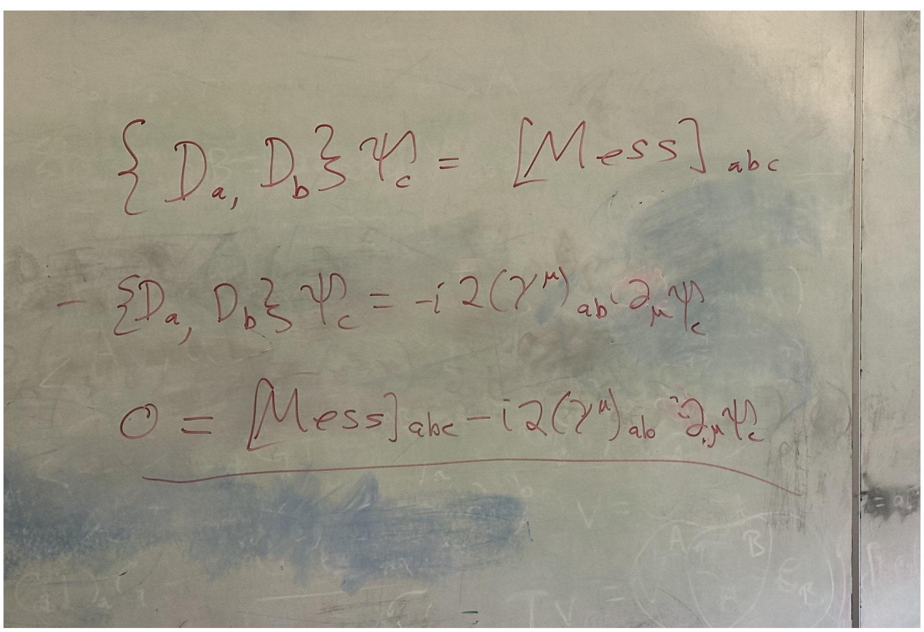
$$(\Sigma_{\mu\nu})_a{}^b = \frac{i}{k_0}[\gamma_\mu, \gamma_\nu]$$

Look back at the homework where we had to find k_0. We also found on the homework that:

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = \eta\Sigma$$

$$[\sigma_{\mu\nu}, \gamma_\rho] = \eta\gamma$$

To summarize, a **supersymmetry algebra** has **supersymmetry generators** whose commutator gives **generators of translation**, with which the supersymmetry generators **commute**. We have this here, with an action with equations of motion that are invariant under the supersymmetry translations. These constraints must hold in general for all fields A, B, ψ, F, G. The A, B terms are relatively easy, the F, G are much harder, and require tricks very similar to the ones used earlier.



The ψ term is more difficult. Using the setup from what Jim wrote, we can raise the b index, and then take the trace of both sides:

$$0 = \delta_b{}^a \left((Mess)_b{}^a - 2i(\gamma^\mu)_a{}^b \partial_\mu \psi_c \right)$$

And so:

$$0 = (Mess)_a{}^a{}_c$$

In doing so, we can expand the matrix in the basis given by the γs and σs (as well as the identity and γ^5. Since we know that our mess is symmetric, we can write it only as a linear combination of γ and σ terms.

20 June 26 — Last Day!

We start by going over the homework problem. Recall that we defined:

$$(Mess)_{abc} = A_{abc} = a_c C_{ab} + b_{\mu c}(\gamma^\mu)_{ab} + c_{\mu c}(\gamma^5\gamma^\mu)_{ab} + d_{\mu\nu c}(\sigma^{\mu\nu})_{ab} + e_c(\gamma^5)_{ab}$$

But we only actually keep the symmetric terms:

$$A_{abc} = b_{\mu c}(\gamma^\mu)_{ab} + d_{\mu\nu c}(\sigma^{\mu\nu})_{ab}$$

Since we know A is symmetric under exchange of the ab indices. We also use the property from last time that:

$$B_c{}^b D_b{}^c = B_{cf} C^{bf} D^{ec} C_{eb} = B_{cf} D^{ec} (\ \delta_e{}^f) = -B_{cb} D^{bc}$$

We can now right multiply by $(\gamma^l)^{ba}$ to get:

$$A_{abc}(\gamma^l)^{ba} = b_{\mu c}(\gamma^\mu)_{ab}(\gamma^l)^{ba} + d_{\mu\nu c}(\sigma^{\mu\nu})_{ab}(\gamma^l)^{ba}$$

But the trace of a $\sigma \cdot \gamma$ term is zero, so:

$$A_{abc}(\gamma^l)^{ba} = b_{\mu c}(\gamma^\mu)_{ab}(\gamma^l)^{ba}$$

And similarly, we can multiply A by $(\sigma^{\mu\nu})^{ba}$ to get:

$$A_{abc} = d_{\mu\nu c}(\sigma^{\mu\nu})_{ab}(\sigma^{kl})^{ba}$$

Explicitly applying those to the expressions we have gives allows us to find the constants $b_{\mu c}$ and $d_{\mu\nu c}$. Once we have these constants, we have written our expression in terms of our desired basis:

$$\{D_a, D_b\}\psi_c = 2i(\gamma^\mu)_{ab}\partial_\mu \psi_c$$

Which then shows that we have the desired closure for the supersymmetric algebra.

Now Konstantinos offers another method:

$$D_{(a}D_{b)}\psi_c = A_{abc}$$

Which is equivalent to the anticommutator. We can instead write:

$$D_{(a}D_{b)} = A_{(ab)c} = \delta_{(a}{}^k\delta_{b)}{}^l A_{klc}$$

And we can recall the Fierz identity from a few days ago:

$$\delta_{(k}{}^\gamma\delta_{\lambda)}{}^\delta = -\frac{1}{2}(\gamma_m)_{\lambda k}(\gamma^m)^{\gamma\delta} - \frac{1}{4}(\sigma_{mn})_{\lambda k}(\sigma^{mn})^{\gamma\delta}$$

And then plug things in to get a result, noting that it will involve some very similar calculations to the above. Using this idea of a basis was not immediately obvious, but significantly organized the calculation for us.

"In math they care about mathematical rigor. In physics — and this is my own personal opinion — I don't care about mathematical rigor, I care about conceptual rigor" — Konstantinos

We can now consider another multiplet besides this one.

"There's a theorist and an experimenter who have offices next door to each other. One day the experimenter runs into the office of the theorist, and says 'look at this data from the experiment I'm doing'. The theorist says 'I think I have a theory that explains that', and starts digging in his drawer for it. But while he is, the experimenter realizes he was showing the data upside down. So when the theorist comes back, the experimenter says 'I'm sorry but I was showing you the data upside down' and he flips it over. And the theorist goes 'I think I have a theory'".

As an alternative, we can look at the **vector supermultiplet**: (A_μ, λ_b, d). The A is our old friend from $E\&M$, but the λ is the partner of the photon, called the photino. Note that it would appear here that we have 5 bosonic degrees of freedom but only 4 fermionic (since ν and a both take 4 different values. But recall that A is only defined up to a gauge transformation:

$$A_\mu \to A_\mu + \partial_\mu\Lambda$$

And so we lose a degree of freedom in A, since we could always pick a gauge such that $A_0 = 0$ (Coulomb gauge), or such that $\partial^\mu A_\mu = 0$ (Lorentz gauge). And so we really only have 3 degrees of freedom in A, and hence we have an appropriate setup for supersymmetry. We also recall that the dimensions of our fields are:

$$[A_\mu] = 1$$
$$[d] = 2$$
$$[\lambda_a] = \frac{3}{2}$$

We want to derive the generators. We can first consider $D_a A_\mu$, remembering that:

$$[D_a] = \frac{1}{2}$$

We know that it must map to a fermion, and it must have appropriate units. We also need to remember that A_μ has a spacetime index, which also must appear on the right. We can do this appropriately by incorporating a γ^μ. And so we can find:

$$D_a A_\mu = a_1(\gamma^\mu)_a{}^b\lambda_b + a_2(\gamma^5\gamma^\mu)_a{}^b\lambda_b$$

We can also try to find $D_a\lambda_b$. In order for dimensions to work, we could map it to some factor times d, but also to a partial derivative of A. Multiplying by $(\gamma^5)_{ab}$ gives a, b indices we need, as does multiplying by C_{ab}. Taking a linear combination of these gives:

$$D_a\lambda_a = a_3(\gamma^5)_{ab}d + a_4C_{ab}\partial^\mu A_\mu + a_5(\gamma^5)_{ab}\partial^\mu A_\mu + a_6C_{ab}d + a_7(\sigma^{\mu\nu})_{ab}\partial_\mu A_\nu$$

However we know that the final term includes the F tensor, which is gauge invariant. The left side is also gauge invariant. So we need the whole right side of the equation to be gauge invariant, and we must kill the terms with A derivatives:

$$D_a\lambda_a = a_3(\gamma^5)_{ab}d + a_6C_{ab}d + a_7(\sigma^{\mu\nu})_{ab}\partial_\mu A_\nu$$

Recall that the Faraday tensor is:

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

Which we can show is gauge invariant by adding the transformation:

$$A_\mu \to A_\mu + \partial_\mu\Lambda$$

And finding that:

$$F_{\mu\nu} \to F_{\mu\nu}$$

Now we need a transformation for d, that is, $D_a d$. We know if must be a fermion, which leaves only lambda, and the dimensions tell us we we need a partial derivative. We can multiply by $(\gamma^\mu)_a{}^b$ to get spacetime indices. We can also add in something that is proportional to $\gamma^5\gamma^\mu$:

$$D_a d = a_8(\gamma^\mu)_a{}^b\partial_\mu\lambda_b + a_9(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b$$

We can also talk about a class of objects called **tensor densities** that include tensors. We can transform a vector as:

$$v^A \to \tilde{v}^A = v^B(T)_B{}^A(\det T)^{h_v}$$

Where h is a number that depends on v. We call this a tensor density. For example, area or volume transforms in this way. So a volume can be described more accurately as a tensor density, and not a tensor.

Some of the transformations we talk about are parity P, time reversal T, and charge conjugation C. We can think about these as tensor densities as well, since some objects are transformed as positive and some as negative. When an object transforms to be negative, we call it "pseudo". This allows us to place additional constraints on some of our terms from before, if we know by assumption if our objects are "pseudo" or not.

Homework problem: Fix the terms in our operators, using the fact that d is a pseudoscalar, A is a vector, and λ a spinor. And recall that A is a gauge object, defined up to a gauge transformation. Also check the closure of the algebra:

$$\{D_a, D_b\} = 2i(\gamma^\mu)\partial_\mu$$

We can also look in the shared folder, at the file called $G1$, and check the closure of some additional supersymmetric algebras.

20.1 Research Problems

The school part of the program is over. We made it. "And now the after party begins. And you don't have to participate in the after party." — Konstantinos

Send Jim an email in the next week if we want to get involved in research. We form research pods, and decide on a pod leader.

Remember that we wrote out the transformations for both the chiral and the vector supermultiplet. These two were some of the earliest supersymmetry discoveries, which were discovered in the 1970s. We did not introduce the idea of superfields. We can introduce a superfield, which has both of the supermultiplets in it. Each of our fields A, B, F, G, D is a function of all four spacetime coordinates. That is, each field maps a point on a 4d manifold to a real number. The A_μ term maps a point to its tangent space, and the fermions are two independent sections of the spin bundle associated with Lorentz transformations.

In a theory with functions of four coordinates, we could consider what might happen if we make things depend on less coordinates. For example, A_μ depends on 4 dimensions. But what would happen if we only looked at 3?. That is:

$$A_\mu \equiv (A_0, A_1, A_2, A_3) \to (A_0, A_1, A_2) \oplus A_3$$

Jim had a professor who said "superfields are not the best way to study supersymmetry". He was very upset, and wanted to prove him wrong. The journey of doing so led him to be recognized as a fellow of the American Mathematical Society. What if we reduce A_μ even further? We can then say:

$$(A_0, A_1, A_2) \oplus A_3 \to (A_0, A_1) \oplus A_3 \oplus A_3$$

So for example, consider:

$$F_{\mu\nu} = \begin{bmatrix} 0 & \partial_0 A_1 - \partial_1 A_0 & \partial_0 A_2 - \partial_2 A_0 & \partial_0 A_3 - \partial_0 A_3 - \partial_3 A_0 \\ \partial_1 A_0 - \partial_0 A_1 & \dots & \ddots & \vdots \\ \vdots & & \vdots & \ddots \\ \dots & \dots & \dots & \vdots \end{bmatrix}$$

And we can see how the field variables still exist but drop out. When you got to projections, the fields don't change, just their arguments. The only difference is that the wave operator only depends on x^0, x^1. After doing a coordinate reduction, we can find that the lagrangians for the chiral and vector multiplets are almost identical. However, the D equations are different. So it looks like we got identical structures with different transformation laws. If we write the quadratic lagrangians they are identical, however if we go to higher power lagrangians that allow for interactions, something interesting happens (recall our W term in the one dimensional case—in any field theory we need at least cubic terms to have forces or interactions). They called this the "twisted chiral multiplet". We can also take this a step further to decrease to one dimension, and look at things that only depend on time. This is where Jim gets Adinkras from, which allows you to study certain systems using graph theory.

The research problem starts with theories in some dimension that represent supersymmetry, and then see what happens when you project them down. See what happens to the quadratic lagrangian. If we find mirror pairs (representations that give the same lagrangian), we see if anything interesting happens when we add in interactions. We start by reducing coordinate dependence, then start looking for representations that look isomorphic. We then examine the transformation laws and see if they are different. Then we look at the lagrangians and see if they look similar or not.

Jim's original starting point was reducing superfields in the literature. We need to find systems that hasn't been studied where the spectra (spin, number, engineering dimension) match.

Big shoutout to the squad:

And of course congrats to all of us for making it to the end:

Thanks everyone for a great program. Have a great rest of the summer, and please stay in touch!